امتصاص عند الطول الموجي426 نانومتر ويتبع قانون بير في مدى التراكيز 1.25 –25 مكغم / مل. وكانت قيمة الامتصاصية المولارية 7177 لتر / مول$^{-1}$.سم$^{-1}$ ودلالة ساندل 0.049 مكغم . سم$^{-2}$ وكانت الطريقة على درجة من الدقة والتوافقية فقد كانت قيمة الاسترجاعية 103.024% وقيمة الانحراف القياسي النسبي ليس اكثر من 1.25 % وقد طبقت الطريقة بنجاح على هيدروكلوريد الكلوربرومازين في المستحضر الصيدلاني(largeactil)

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Best Simultaneous Approximation of Bounded Functions via Linear Operators in $\mathcal{N}_2$

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Abstract

The purpose of the current work is to present the concept of proximinal simultaneous approximation, and to prove the continuous of operator best simultaneous approximation in two-normed spaces. We try to find the best simultaneous approximation of bounded function by using linear operators in two- normed space.

Keywords: Two-normed spaces, simultaneous proximinal and continuous simultaneously operator.

1. INTRODUCTION

The notes of the best approxima has been principal presented via I. Singer(1974), in Boszny discussed set of remark on simultaneous approximation (1978) [2] , Li and Watson proved a set of results about best simultaneous approximation (1996) & (1997) respectively [3], [4], then Boyd and Vandenberghe explained characteristic of convex set (2004) [9], and Mohebi achieved some properties of best simultaneous approximation (2005) [5]. Abu-sirhan presented a set of researches that include the best simultaneous approximation in $L^{\infty}(I,X)$, operator and functional spaces (2009) & (2012) respectively [6], [7], [8] , in the end ,I benefited greatly from the papers of the tow scholars Elumalai and Makandeya, on best simultaneous approximation (2009) & (2013) [10,11]. In the present work, we studied some marks of simultane. approxima. of bounded mappings by using linear operators in two-normed space. These are the results which are proven in 2-normed space; our main results are continuous and we find best simultaneous approximation set in two-normed space.

2. PRELIMINARIES

Definition 2.1 : $\mathcal{N}_2 : \mathcal{H} \times \mathcal{H} \to \mathcal{R}$ be real mapping which gratify the following properties

1) For $v_0, v_1 \in \mathcal{H}$, $\|v_0, v_1\|_{b,2} = 0$ if and only if v_0, v_1 are linearly dependent .

2) $\|v_0, v_1\|_{b,2} = \|v_1, v_0\|_{b,2}$,

3) $\|av_0, v_1\|_{b,2} = |a| \, \|v_0, v_1\|_{b,2}$,

4) $\|v_0, v_1 - v_2\|_{b,2} \leq \|v_0, v_1 - v_3\|_{b,2} + \|v_0, v_3 - v_2\|_{b,2}$, for every $v_0, v_1, v_2, v_3 \in \mathcal{H}$ and $a \in \mathbb{R}$ then $\mathcal{N}_2$ is named two − normed linear space .

Definition 2.2: Let $\mathcal{N}_2$ be a two − normed linear space . for $\phi \neq \mathcal{B}, \mathcal{M} \subseteq \mathcal{N}_2$,

d$(\mathcal{B}, \mathcal{M}) = sup_{b \in \mathcal{B}}\{\|v_0, v_1 - m\|_{b,2}\}$ $m \in \mathcal{M}$, denotes the distance

from the set $\mathcal{B}$ to the set $\mathcal{M}$. If

$sup_{b \in \mathcal{B}}\{\|v_0, v_1 - m\|_{b,2}\} = sup_{b \in \mathcal{B}}\{\|v_0, v_1 - m_0\|_{b,2}\}$.

Then, we say that a function $m_0 \in \mathcal{M}$ is named a better approximation from $\mathcal{B}$ to $\mathcal{M}$.

Definition 2.3: Let $\mathcal{N}_2$ be a two − normed linear space on $\mathbb{F}$ ($real\ field$), $\mathcal{M}_a$ subspace of $\mathcal{N}_2$ and ($M \neq \emptyset$) a subset of $\mathcal{M}_a$. for a bounded subset $\mathcal{Y}$ of $\mathcal{N}_2$. As define $\mathcal{N}_2$

$rad_M(\mathcal{Y}) = inf_{p \in M} \, sup_{a \in \mathcal{Y}} \|z , a - p\|_{b,2}$

for $\in \mathcal{N}_2 / \mathcal{M}_a$, and $cent_M(\mathcal{Y}) = p_1 \in M$:

$sup_{a \in \mathcal{Y}} \|z, a - P_1\|_{b,2} = rad_M(\mathcal{Y})$ for every $z \in \mathcal{N}_2 / \mathcal{M}_a$.

The number $rad_M(\mathcal{Y})$ is called the Chebyshev radius of $\mathcal{Y}$ with respect to M and an element $p_1 \in cent_M(\mathcal{Y})$ is called a better simultaneous approximation.

Definition 2.4. [9]**:** A subspace $\mathcal{M}$ of X (vector space) is called convex if $m_1, m_2 \in \mathcal{M}$ implies

D= $[w \in X,\ w = j m_1 + (1-j) m_2,\ 0 \leq j \leq 1] \subset \mathcal{M}$.

Definition 2.5: A 2-normed ν is said to be continuous at (p,q) if

for a given $\in > 0$ there $\exists$ a $i > 0$ $\ni$

$|\nu(p,q) - \nu(w,d)| < \in$ whenever $\|p - w, q\|_{b,2} < i$ &

$\|w, q - d\|_{b,2} < i$ or $\|p - w, d\|_{b,2} < i$ and $\|p, q - d\|_{b,2}$.

Then ν is said to be continuous at each point the domain .

3. Auxiliary lemmas

Lemma 3.1: Let $\mathcal{N}_2$ be two − normed linear space and $\mathcal{M}, \mathcal{B}$ be closed sub space of $\mathcal{N}_2$. Then

$\|n, m \oplus b\|_{b,2} = \|n, m\|_{b,2} \oplus \|n, b\|_{b,2}$

from $n \in \mathcal{N}_2$ and $m \in \mathcal{M}$, b $\in \mathcal{B}$.

Proof: For $\nu \in \|n, m \oplus b\|_{b,2}$, let $\nu_0 : \mathcal{N}_2 \longrightarrow \mathcal{M}$ and $\nu_1 : \mathcal{N}_2 \longrightarrow \mathcal{B}$ be such that $\nu(u) = \|\nu_0(u), \nu_1(u) \oplus m\|_{b,2}$ for all $u \in \mathcal{N}_2$, $m \in \mathcal{M}$.

It is clear that $\nu_0 \in \|n, m\|_{b,2}$ and $\nu_1 \in \|n, b\|_{b,2}$. Define

$\varphi : \|n, m \oplus b\|_{b,2} \longrightarrow \|n, m\|_{b,2} \oplus \|n, b\|_{b,2}$

from $n \in \mathcal{N}_2$ and $m \in \mathcal{M}$, b $\in \mathcal{B}$, by

$\varphi(\nu) = \|\nu_0, \nu_1 \oplus m\|_{b,2}$. It is clear that φ is onto isometry, noting that

$\|\varphi(\nu)\|_{b,2}$ = max $\{\|\nu_0, \nu_1 \oplus m\|_{b,2}\}$

= sup max $\{\|\nu_0(u), \nu_1(u) \oplus m\|_{b,2}\}$

= sup $\|\nu(u)\|_{b,2} = \|\nu\|_{b,2}$.

4. Main results

Theorem 4.1: Let $\mathcal{M}$ be a closed sub space of two − normed linear space $\mathcal{N}_2$, for any $\nu_0, \nu_1 \in \mathcal{N}_2$, we have

$\|v_0(u), v_1(u) - w\|_{b,2} \leq \|v_0, v_1 - w\|_{b,2}\ w \in \mathcal{M}.$

Proof: Since $\mathcal{M} \subseteq \mathcal{N}_2$, $w \in \mathcal{M}$, we need to proof

$\|v_0(u), v_1(u) - w\|_{b,2} \leq \|v_0, v_1 - w\|_{b,2}\ w \in \mathcal{M}.$

$\|v_0(u), v_1(u) - w\|_{b,2} \leq \max \|v_0(u), v_1(u) - w\|_{b,2}$,

$\|v_1(u), v_0(u) - w\|_{b,2}$. We take super. two sides , we obtain

$sup\ \|v_0(u), v_1(u) - w\|_{b,2} \leq \sup\ \max\ \{\|v_0, v_1 - w\|_{b,2}, \|v_1, v_0 - w\|_{b,2}\}$. Since $w \in \mathcal{M}$ was arbitrary , then

$sup\ \|v_0(u), v_1(u) - w\|_{b,2} \leq \|v_0, v_1 - w\|_{b,2}$.

Now, let $j > \sup \|v_0(u), v_1(u) - w\|_{b,2}$ for $u \in \mathcal{N}_2$, define

$$\varphi(u) = \{m \in \mathcal{M} : \max\{\|v_0(u), v_1(u) - m\|_{b,2}, \|v_1(u), v_0(u) - m\|_{b,2}\} \leq \sup \|v_0(u), v_1(u) - w\|_{b,2}\}.$$

The $\emptyset \neq \varphi$ is $\underbrace{subset\ of\ \mathcal{M}}_{closed}$. Now , to show that

$\varphi(u)$ is convex for $\forall\ u \in \mathcal{N}_2$ and φ is lower semi continuous .

Let $\in \mathcal{N}_2$, $m_1, m_2 \in \varphi(u)$, and $0 \leq \delta \leq 1$.

$$\max\{\|v_0(u), v_1(u) - \delta m_1 - (1-\delta)m_2\|_{b,2}, \|v_1(u), v_0(u) - \delta m_1 - (1-\delta)m_2\|_{b,2}\}$$

$$\leq \max \{\delta \|v_0(u), v_1(u) - m_1\|_{b,2} + (1-\delta)\|v_0(u), v_1(u) - m_2\|_{b,2}, \delta\|v_1(u), v_0(u) - m_1\|_{b,2} + (1-\delta)\|v_1(u), v_0(u) - m_2\|_{b,2}\}$$

$$\leq \delta \max\{\|v_0(u), v_1(u) - m_1\|_{b,2}, \|v_1(u), v_0(u) - m_1\|_{b,2}\}$$

$$+ (1-\delta)\max\{\|v_0(u), v_1(u) - m_2\|_{b,2}, \|v_1(u), v_0(u) - m_2\|_{b,2}\}$$

$$\leq \delta \sup \|v_0(u), v_1(u) - w\|_{b,2} + (1-\delta)\sup\|v_0(u), v_1(u) - w\|_{b,2} = \sup \|v_0(u), v_1(u) - w\|_{b,2}.$$

To demonstration that φ is lower semi continuous , let p be an open set in $\mathcal{M}$.

to investeg ate $p^* = \{u \in \mathcal{N}_2 : \varphi(u) \cap p \neq \emptyset\}$.

It is to be exposed p^* is open . Let $e \in p^*$, then $\varphi(e) \cap p \neq \emptyset$. Hance , there exists an $m \in p$ such that

$$\max\{\|v_0(e), v_1(e) - m\|_{b,2}, \|v_1(e), v_0(e) - m\|_{b,2}\} \leq \sup \|v_0(u), v_1(u) - w\|_{b,2}$$

By the definition of sup $\|v_0(u), v_1(u) - w\|_{b,2}$, sup $\|v_0(u), v_1(u) - w\|_{b,2} > inf_{q \in \mathcal{M}}$ max $\{\|v_0(e), v_1(e) - q\|_{b,2}, \|v_1(e), v_0(e) - q\|_{b,2}\}$, there exists $m' \in \mathcal{M}$ such that

max $\left\{\left\|v_0(e), v_1(e) - m'\right\|_{b,2}, \left\|v_1(e), v_0(e) - m'\right\|_{b,2}\right\}$

$< \sup \|v_0(u), v_1(u) - w\|_{b,2}$.

Now, $m \in p$, then there exists $\epsilon > 0$ such that

$D(m, \epsilon) = \{q \in \mathcal{M} : \|q - m\|_{b,2} < \epsilon\} \subseteq p$.

Let $i = \frac{\epsilon}{2\|m - m'\|_{b,2}}$ if $\|m - m'\|_{b,2} \geq 1 \frac{\epsilon}{2}$ if $\|m - m'\|_{b,2} \leq 1$;

consider that $0 \leq i \leq 1$. Let $m'' = (1 - i)m + i\,m'$, then

$\|m'' - m\|_{b,2} = i\,\|m - m'\|_{b,2} < \epsilon$, hance $m'' \in p$.

By the covexity of $\varphi(e)$, $m'' \in \varphi(e)$ and

max $\left\{\left\|v_0(e), v_1(e) - m''\right\|_{b,2}, \left\|v_1(e), v_0(e) - m''\right\|_{b,2}\right\}$

$< \sup \|v_0(u), v_1(u) - w\|_{b,2}$

Now, let L be open ball of e such that

max $\{\|v_0(e), v_1(e) - v_1(u)\|_{b,2}, \|v_1(e), v_0(e) - v_0(u)\|_{b,2}\}$ $< \sup \|v_0(u), v_1(u) - w\|_{b,2}$ $- max\left\{\left\|v_0(e), v_1(e) - m''\right\|_{b,2}, \left\|v_1(e), v_0(e) - m''\right\|_{b,2}\right\}$

For any $u \in L$, we have

max $\left\{\left\|v_0(u), v_1(u) - m''\right\|_{b,2}, \left\|v_1(u), v_0(u) - m''\right\|_{b,2}\right\}$

$\leq$max $\left\{\|v_0(u), v_1(u) - v_1(e)\|_{b,2} + \left\|v_0(e), v_1(e) - m''\right\|_{b,2},\ \|v_1(u), v_0(u) - v_0(e)\|_{b,2} + \left\|v_1(e), v_0(e) - m''\right\|_{b,2}\right\}$

$\leq$ max $\{\|v_0(u), v_1(u) - v_1(e)\|_{b,2}, \|v_1(u), v_0(u) - v_0(e)\|_{b,2}\}$ + max $\left\{\left\|v_0(e), v_1(e) - m''\right\|_{b,2}, \left\|v_1(e), v_0(e) - m''\right\|_{b,2}\right\}$

$\leq \sup \|v_0(u), v_1(u) - w\|_{b,2}$.

Hence, $m'' \in \varphi(u) \cap p$, $u \in p^*$, L $\in p^*$ and p is open .

There exists $w \in \mathcal{M}$ such that $w(u) \in \varphi(u)$ for all $u \in \mathcal{N}_2$. Hance max $\left\{\left\|v_0(u),v_1(u)-w(u)\right\|_{b,2},\|v_1(u),v_0(u)-w(u)\|_{b,2}\right\}$

$\leq$ sup $\|v_0(u),v_1(u)-w\|_{b,2}$ and max $\left\{\left\|v_0,v_1-w\right\|_{b,2},\|v_1,v_0-w\|_{b,2}\right\}\leq$ sup $\|v_0(u),v_1(u)-w\|_{b,2}$.

Thus, $\|v_0,v_1-w\|_{b,2}\leq sup\|v_0(u),v_1(u)-w\|_{b,2}$, $w\in\mathcal{M}$.

Theorem 4.2: Let $\mathcal{N}_2$ be two − normed linear space and $\mathcal{M}$ be a closed subspace of $\mathcal{N}_2$. Then

1. If $\|n,m\|_{b,2}$ from $n\in\mathcal{N}_2$ and $m\in\mathcal{M}$ is simultaneous. proximinal in $\mathcal{N}_2$, then $\mathcal{M}$ simultaneous. proximinal in $\mathcal{N}_2$.
2. If $\mathcal{M}$ has a continuous simultaneous. operator , then $\|n,m\|_{b,2}$ from $n\in\mathcal{N}_2$, $m\in\mathcal{M}$ is simultaneous. proximinal in $\mathcal{N}_2$ and has continuous simultaneous. proximity operator .

Proof: 1. Let $x_1,y_1\in\mathcal{N}_2$. Define $v_{y_1}:\mathcal{N}_2\longrightarrow\mathcal{N}_2$ and

$v_{x_1}:\mathcal{N}_2\longrightarrow\mathcal{N}_2$ by $v_{y_1}(u)=y_1$, $v_{x_1}(u)=x_1$ for all $u\in\mathcal{N}_2$.

Since $\|n,m\|_{b,2}$ from $n\in\mathcal{N}_2$ and $m\in\mathcal{M}$ is simultaneous.

proximinal in $\mathcal{N}_2$, $\exists\, g\in\|n,m\|_{b,2}$ such that ,

max $\left\{\left\|v_{y_0},v_{y_1}-g\right\|_{b,2},\left\|v_{x_0},v_{x_1}-g\right\|_{b,2}\right\}=$ sup $\left\|v_{y_1},v_{x_1}-m\right\|_{b,2}$

$=$ sup $\left\|v_{x_1},v_{y_1}-m\right\|_{b,2}\leq\|x_1,y_1-m\|_{b,2}$.

Then , for some $u_o\in\mathcal{N}_2$, we have

max $\left\{\left\|v_{y_0}(u_o),v_{y_1}(u_o)-g(u_o)\right\|_{b,2},\left\|v_{x_0}(u_o),v_{x_1}(u_o)-g(u_o)\right\|_{b,2}\right\}\leq\|x_1,y_1-m\|_{b,2}$. Hence $g(u_o)$ is a better simultaneo.

approx. for x_1,y_1 of $\mathcal{M}$.

2. Let $\mathcal{B}:\mathcal{N}_2\oplus\mathcal{N}_2\longrightarrow\mathcal{M}$ be a cont's simultaneous.

proximity operator for $\mathcal{M}$. Define

$\mathcal{B}':\|n,n\oplus n\|_{b,2}\longrightarrow\|n,m\|_{b,2}$ from $n\in\mathcal{N}_2$ and $m\in\mathcal{M}$,

by $\mathcal{B}'(v)=\mathcal{B}\circ v$. $\mathcal{B}'$ can be redefined as

$\|n,n\|_{b,2}\oplus\|n,n\|_{b,2}\longrightarrow\|n,m\|_{b,2}$, and

$\mathcal{B}\|v_0,v_1-m\|_{b,2}$ for all $m\in\mathcal{M}$. It is clear that

$\mathcal{B}'\|v_0,v_1-m\|_{b,2}\in\|n,m\|_{b,2}$. Let $g\in\|n,m\|_{b,2}$, then max $\{\|v_0, v_1 - \mathcal{B}(v_1)\|_{b,2}, \|v_0, v_0 - \mathcal{B}(v_0)\|_{b,2}\}$

$\leq$ max $\{\|v_0, v_1 - g\|_{b,2}, \|v_0, v_o - g\|_{b,2}\}$.

Thus , $\mathcal{B}\ \|v_0, v_1 - g\|_{b,2}$ is a better simultaneo. for v_0, v_1

from $\|n, m\|_{b,2}$ and then $\|n, m\|_{b,2}$ is simultaneous. proximinal

in $\mathcal{N}_2$. It is clear that $\mathcal{B}\ \acute{}: \mathcal{N}_2 \oplus \mathcal{N}_2 \longrightarrow \mathcal{M}$ is a continuous

simultaneo. proximity operator .

Theorem 4.3: Let $\mathcal{M}$ be a simultaneo. sub space of a two − normed linear space $\mathcal{N}_2$. If $\mathcal{M}$ has a linear proximity operator ,

then $\| z, m \|_{b,2}$ is simultaneous. proximinal in $\| z, n \|_{b,2}$

from $z \in \mathcal{N}_2$ and $m \in \mathcal{M}$, $n \in \mathcal{N}_2$ and has a linear simultaneo. proximity operator .

Proof: Let $\varphi : \mathcal{N}_2 \oplus \mathcal{N}_2 \longrightarrow \mathcal{M}$ be a linear simultaneous. proximity operator for $\mathcal{M}$. Define anther operator

$\mathcal{B} : \|z, n \oplus n\|_{b,2} \longrightarrow \|z, m\|_{b,2}$ from $z \in \mathcal{N}_2$ and $m \in \mathcal{M}$, $n \in \mathcal{N}_2$,

given $\mathcal{B}(v) = \varphi \circ v$. we may write $\mathcal{B} : \|z, n\|_{b,2} \oplus \|z, n\|_{b,2} \longrightarrow \|z, m\|_{b,2}$, define by $\|v_0, v_1 - m\|_{b,2} = \varphi \circ \|v_0, v_1 - m\|_{b,2}$.

Since $\mathcal{B}$ is linear operator, we have

$\mathcal{B}\{\delta \| v_0, v_1 - m\|_{b,2} + \lambda \| g_0, g_1 - m\|_{b,2}\}$

$= \delta\, \mathcal{B} \| v_0, v_1 - m\|_{b,2} + \lambda\, \mathcal{B} \| g_0, g_1 - m\|_{b,2}$, for all $m \in \mathcal{M}$

and $\|v_0, v_1 - m\|_{b,2}$ is a linear simultaneous. proximity operator

for $\| z, m \|_{b,2}$.

Theorem 4.4: Let $\mathcal{N}_2$ be two − normed linear space , $\mathcal{M}$ closed

sub space of $\mathcal{N}_2$ and $\emptyset \neq Y \subseteq \mathcal{N}_2$. Then $\mathcal{M}$ is simultaneo. proximinal in $\mathcal{N}_2$ if and only if $\| y, m \|_{b,2}$ is simultaneo. proximinal

in $\| y, n \|_{b,2}$ for y $\in Y$ and $m \in \mathcal{M}$, $n \in \mathcal{N}_2$.

Proof: Assume that $\mathcal{M}$ is simultaneo. proximinal in $\mathcal{N}_2$,

we need to prove $\| y, m \|_{b,2}$ is simultaneo. proximinal

in $\| y, n \|_{b,2}$ for y $\in Y$ and $m \in \mathcal{M}$, $n \in \mathcal{N}_2$.

Let $v, g \in \| y, n \|_{b,2}$ for all y $\in Y$ and $n \in \mathcal{N}_2$. Since $\mathcal{M}$ is simultaneo. proximinal in $\mathcal{N}_2$, the for any $u \in Y$ there exists

T$(u) \in \mathcal{M}$ such that

$$\max\left\{\left\|v_0(u), v_1(u) - T(u)\right\|_{b,2}, \|g_0(u), g_1(u) - T(u)\|_{b,2}\right\}$$

$$\leq \max\left\{\left\|v_0(u), v_1(u) - w\right\|_{b,2}, \|g_0(u), g_1(u) - w\|_{b,2}\right\},$$

for all $w \in G$. In particular it holds for any $(u) \in G$,

$w \in \| y, m \|_{b,2}$. By Axiom of choice , the exists T $\in \| y, m \|_{b,2}$.

Hence $\max\left\{\left\|v_0, v_1 - T\right\|_{b,2}, \|g_0, g_1 - T\|_{b,2}\right\}$

$\leq \max\left\{\left\|v_0, v_1 - w\right\|_{b,2}, \|g_0, g_1 - w\|_{b,2}\right\}$, for all $w \in \| y, m \|_{b,2}$.

Then $\max\left\{\left\|v_0, v_1 - T\right\|_{b,2}, \|g_0, g_1 - T\|_{b,2}\right\} = \|v, g - m\|_{b,2}$,

which implies $\| y, m \|_{b,2}$ is simultaneous. proximinal

in $\| y, n \|_{b,2}$ for all y $\in Y$ and $m \in \mathcal{M}$, $n \in \mathcal{N}_2$.

Converse : Assume $\| y, m \|_{b,2}$ is simultaneo. proximinal

in $\| y, n \|_{b,2}$ for y $\in Y$ and $m \in \mathcal{M}$, $n \in \mathcal{N}_2$, we need to prove

$\mathcal{M}$ is simultaneo. proximinal in $\mathcal{N}_2$.

Let $x_1, y_1 \in \mathcal{N}_2$. Set $v_{x_1} : Y \longrightarrow \mathcal{N}_2$, $v_{y_1} : Y \longrightarrow \mathcal{N}_2$,

define by $v_{x_1}(u) = x_1$, $v_{y_1}(u) = y_1$ for all $u \in Y$.

$\left\|v_{y_1}, v_{x_1} - m\right\|_{b,2} = \sup \left\|v_{y_1}(u), v_{x_1}(u) - m\right\|_{b,2} = \|x_1, y_1 - m\|_{b,2}$

for all $m \in \mathcal{M}$. Since $\| y, m \|_{b,2}$ for all y $\in Y$ and $m \in \mathcal{M}$

is simultaneous. proximinal in $\| y, n \|_{b,2}$ for all y $\in Y$ and $n \in \mathcal{N}_2$,

then there exists $g \in \| y, m \|_{b,2}$ such that

$$\max\left\{\left\|v_{x_0}, v_{x_1} - g\right\|_{b,2}, \left\|v_{y_0}, v_{y_1} - g\right\|_{b,2}\right\} = \|x_1, y_1 - m\|_{b,2}$$

$\forall\, m \in \mathcal{M}$. Choose $u_0 \in Y$ such that

$$\max\left\{\|x_0, x_1 - g(u_0)\|_{b,2}, \|y_0, y_1 - g(u_0)\|_{b,2}\right\}$$

$\leq \|x_1, y_1 - w\|_{b,2}$ for all $w \in G$. Then $g(u_0)$ is a better

simultaneo. approx. for x_1 and y_1 form $\mathcal{M}$.