Some Geometric Properties on a Subclass of Meromorphic Multivalent Functions Associated with Integral Operator

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Abstract: In this paper, we have defined a subclass $\mathcal{M}_q^{h,x}(\delta, G, D)$, of meromorphic multivalent functions by using integral operator $\mathfrak{Z}_p^x f(z)$. Also, we have obtained the coefficient bounds for the function of the form $f(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} a_{h-q} b_{h-q} z^{h-q}$ $(a_{h-q} \geq 0, q \in \mathbb{N})$, and belong to the class $\mathcal{M}_q^{h,x}(\delta, G, D)$. We have get some important geometric properties of coefficient estimates such as extreme points and we have proved that convex linear combination is in the same class. A growth and distortion theorems were introduced. Furthermore, we have deduced the radii of starlikeness and convexity theorems and partial sum property is defined. At last, in the present paper some concepts like neighborhood for analytic Univalent functions are going to be introduced and proved that $N_\epsilon(f) \subset \mathcal{M}_q^{h,x}(\delta, G, D)$.

Keyword: Meromorphic multivalent functions, integral operator, extreme points, growth and distortion bounds, ϵ-neighborhood property, partial sum property.

المستخلص: في ورقة البحث هذه قمنا بتعريف فئة فرعية $\mathcal{M}_q^{h,x}(\delta, G, D)$ للدالة الميرومورفيه المتعدده باستخدام مؤثر التكامل $\mathfrak{Z}_p^x f(z)$. كذلك حصلنا على حدود المعاملات للدالة التي من الصيغة $f(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} a_{h-q} b_{h-q} z^{h-q}$ $(a_{h-q} \geq 0, q \in \mathbb{N})$, والتي تنتمي الى الفئة $\mathcal{M}_q^{h,x}(\delta, G, D)$. لقد قمنا بالحصول على بعض الخواص الهندسية المهمة لتقديرات المعامل مثال ذلك النقاط المتطرفة وقمنا ببرهان التركيب الخطي المحدب في نفس الفئة. نظريات النمو والتشويه قد قدمت ايضاً. بالإضافة الى ذلك قمنا باستنتاج نظريات انصاف الاقطار النجمية والمحدبة وعرفنّا خاصية الجمع الجزئي. اخيراً في الورقة البحثية

المقدمة هذه بعض المبادئ كالجوار للدوال التحليلية الاحادية التكافؤ قد قدمت وبُرهنت وذلك بأن
$\cdot N_\epsilon(f) \subset \mathcal{M}_q^{h,x}(\delta, G, D)$

1.Introduction

Meromorphic multivalent have been studied by many authors in field differential geometry [1-5] and mathematical finance [6-10]

Let $\mathcal{M}_q^h$ denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} a_{h-q} z^{h-q} \ ,\left(a_{h-q} \geq 0, \ \ q \in \mathbb{N}\right), \qquad (1.1)$$

which analyze in the open, pierced disk

$U^* = \{z: z \in \mathbb{C}, 0 < |z| < 1\}$. If $f \in \mathcal{M}_q^h$

Presented by (1.1), and

$$g(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} b_{h-q} z^{h-q} \ ,(b_{h-q} \geq 0, \ \ q \in \mathbb{N})$$

then the convolution $f * g$ defined by

$$(f * g)(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} a_{h-q} b_{h-q} z^{h-q} = (g * f)(z).$$

For the function $f \in \mathcal{M}_q^h$, we define the integral operator Seoudy [11] $\mathfrak{Z}_q^x f(z): \mathcal{M}_q^h \to \mathcal{M}_q^h$ as 01

$\mathfrak{Z}_q^x f(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} \left[\frac{010}{h-q+1}\right]^x a_{h-q} z^{h-q} \ ,(x \geq 0, \ \ q \in \mathbb{N})$.Which studied by [12].

2. Definition: [13] We apply $\mathfrak{Z}_q^x f(z)$ to define the class $\mathcal{M}_q^h$ as follow. The function provided by (1.1) meets the inequality, it is said to be in the class $\mathcal{M}_q^{h,x}(\delta, G, D)$

$$\left| \frac{\delta z^{q+2}\left[\mathfrak{Z}_q^x f(z)\right]'' + z^{q+1}\left[\mathfrak{Z}_q^x f(z)\right]' - q[\delta(q+1) - 1]}{D\left[\delta z^{q+2}\left[\mathfrak{Z}_q^x f(z)\right]'' + z^{q+1}\left[\mathfrak{Z}_q^x f(z)\right]'\right] - Gq[\delta(q+1) - 1]} \right| < 1, \qquad (2.1)$$

where $\left(z \in U^*, \varkappa \geq 0,, \ q \in \mathbb{N}, -1 \leq D < G \leq 1, \ 0 \leq \delta < \frac{1}{q+1}\right)$.

3. Main Results

The coefficient constraints for the function of the from (1.1) to belong to the class $\mathcal{M}_q^{h,\varkappa}(\delta, G, D)$ are as follows.

Theorem (3.1): Suppose that $f \in \mathcal{M}_q^h$, the function $f \in \mathcal{M}_q^{h,\varkappa}(\delta, G, D)$ if and only if

$$\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{\varkappa}(1-D)(h-q)[\delta(h-q-1)+1]a_{h-q}$$

$$\leq q(D-G)[\delta(q+1)-1], \qquad (3.1)$$

the inequality is sharp for $f(z)$ given by

$$f(z) = \frac{1}{z^q} + \frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{\varkappa}(1-D)(h-q)[\delta(h-q-1)+1]} z^{h-q}.(h \geq 1)$$

Proof: Let the inequality (3.1) holds true and assume that $0 < |z| = r < 1$. Then from (2.1) we get

$$\mathcal{A}_f = \left|\delta z^{q+2}[\mathfrak{Z}_q^{\varkappa} f(z)]'' + z^{q+1}[\mathfrak{Z}_q^{\varkappa} f(z)]' - q[\delta(q+1)-1]\right|$$

$$-\left|D\left[\delta z^{q+2}[\mathfrak{Z}_q^{\varkappa} f(z)]'' + z^{q+1}[\mathfrak{Z}_q^{\varkappa} f(z)]'\right] - Gq[\delta(0q+1)-1]\right|$$

$$= \left|\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{\varkappa}(h-q)[\delta(h-q-1)+1]a_{h-q}z^h\right|$$

$$-\left|q(D-G)[\delta(q+1)-1] + D\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{\varkappa}(h-q)[\delta(h-q-1)+1]a_{h-q}z^h\right|$$

$$\leq \sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{\varkappa}(h-q)[\delta(h-q-1)+1]a_{h-q}r^h$$

$$-q(D-G)[\delta(q+1)-1] - q(D-G)[\delta(q+1)-1]$$

$$-D\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(h-q)[\delta(h-q-1)+1]a_{h-q}r^{h}$$

$$\leq \sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]a_{h-q}$$

$$-q(D-G)[\delta(q+1)-1]\leq 0.$$

From (3.1), hence $f\in\mathcal{M}_q^{h,x}(\delta,G,D)$.

Conversely, assume that $f\in\mathcal{M}_q^{h,x}(\delta,G,D)$,then

$$\left|\frac{\delta z^{q+2}[\mathfrak{Z}_q^{x}f(z)]''+z^{q+1}[\mathfrak{Z}_q^{x}f(z)]'-q[\delta(q+1)-1]}{D\left[\delta z^{q+2}[\mathfrak{Z}_q^{x}f(z)]''+z^{q+1}[\mathfrak{Z}_q^{x}f(z)]'\right]-Gq[\delta(q+1)-1]}\right|$$

$$=\left|\frac{\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(h-q)[\delta(h-q-1)+1]a_{h-q}z^{h}}{q(D-G)[\delta(q+1)-1]+D\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(h-q)[\delta(h-q-1)+1]a_{h-q}z^{h}}\right|<1,$$

since $Re(z)\leq|z|$ for all z,

$$Re\left(\frac{\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(h-q)[\delta(h-q-1)+1]a_{h-q}z^{h}}{q(D-G)[\delta(q+1)-1]+D\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(h-q0)[\delta(h-q-1)+1]a_{h-q}z^{h}}\right)<1.$$

By letting $z\to 1^{-}$, we have

$$\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(1-D)(n-q)[\delta(h-q-1)+1]a_{h-q}$$

$$\leq q(D-G)[\delta(q+1)-1].$$

Last but not least, if we take.

$$f(z)=\frac{1}{z^{q}}+\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]}z^{h-q}.(h\geq 1).$$

Hence, the proof is complete.

Corollary (3.2): If $f\in\mathcal{M}_q^{h,x}(\delta,G,D)$, then

$$a_{h-q} \leq \frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]} z^{h-q}, (h \geq 1)$$

where $\left(z \in U^*, x \geq 0,, \ q \in \mathbb{N}, -1 \leq D < G \leq 1, \ 0 \leq \delta < \frac{1}{1+q}\right)$.

Corollary (3.3): If $0 \leq \delta_1 < \delta_2 < \frac{1}{q+1}$, then

$$\mathcal{M}_q^{h,x}(\delta_1, G, D) \subset \mathcal{M}_q^{h,x}(\delta_2, G, D).$$

In the following theorem, we get the extreme points of the class $\mathcal{M}_q^{h,x}(\delta, G, D)$

Theorem (3.4): The function $f \in \mathcal{M}_q^{h,x}(\delta, G, D)$ given by (1.1) if and only if the conditions below are true

$$f(z) = \sum_{h=0}^{\infty} c_{h-q} f_{h-q}(z), c_{h-q} \geq 0, \ \sum_{h=0}^{\infty} c_{h-q} = 1,$$

where

$$f_{-q}(z) = \frac{1}{z^q},$$

$$f_{-q}(z) = \frac{1}{z^q} + \frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]} z^{h-q}. (h \geq 1)$$

Proof: Assume that

$$f(z) = \sum_{h=0}^{\infty} c_{h-q} f_{h-q}(z), c_{h-q} \geq 0, \ \sum_{h=0}^{\infty} c_{h-q} = 1,$$

then

$$f(z) = c_{-q} f_{-q}(z) + \sum_{h=1}^{\infty} c_{h-q} \left[\frac{1}{z^q} + \frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]} z^{h-q}\right]$$

$$+ \sum_{h=1}^{\infty} c_{h-q} \left[\frac{1}{z^q} + \frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]} z^{h-q}\right]$$

$$= \frac{1}{z^q} + \sum_{h=1}^{\infty} \frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^x (1-D)(h-q)[\delta(h-q0-01)+1]} c_{h-q} z^{h-q} .$$

From Theorem (3.1), we have $f \in \mathcal{M}_q^{h,x}(\delta, G, D)$.

Conversely, assume that $f \in \mathcal{M}_q^{h,x}(\delta, G, D)$ and by letting $c_{-q} = 1 - \sum_{h=1}^{\infty} c_{h-q}$, where

$c_h = \frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^x (1-D)(h-q)[\delta(h-q-1)+1]} a_{h-q}$, this concludes the results.

Theorem (3.5): Suppose that

$$f_\tau(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} a_{h-q,\tau} z^{h-q} \quad , \tau = 1,2,3, \dots , s$$

and $f_\tau(z) \in \mathcal{M}_q^{h,x}(\delta, G, D)$, then $f(z) = \sum_{\tau=1}^{s} c_{\tau-q} f_\tau(z)$ in the same class where $\sum_{\tau=1}^{s} c_{\tau-q} = 1$.

***Proof*:** Theorem (3.1) states that we have

$$\sum_{h=1}^{\infty} \left[\frac{1}{h+q+1}\right]^x (1-D)(h-q)[\delta(h-q-1)+1] a_{h-q,\tau}$$

$$\leq q(D-G)[\delta(q+1)-1] (\tau = 1,2,3, \dots , s).$$

But

$$f(z) = \sum_{\tau=1}^{s} c_{\tau-q} f_\tau(z) = \sum_{\tau=1}^{s} c_{\tau-q} \left[\frac{1}{z^q} + \sum_{h=1}^{\infty} a_{h-q,\tau} z^{h-q} \right]$$

$$= \frac{1}{z^q} \sum_{\tau=1}^{s} c_{\tau-q} + \sum_{h=1}^{\infty} \left[\sum_{\tau=1}^{s} c_{\tau-q} a_{h-q,\tau}\right] z^{h-q}$$

$$= \frac{1}{z^q} + \sum_{h=1}^{\infty} \left[\sum_{\tau=1}^{s} c_{\tau-q} a_{h-q,\tau}\right] z^{\tau-q},$$

since

$$\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1][\sum_{\tau=1}^{s}c_{\tau-q}\, a_{h-q,\tau}]$$

$$=\sum_{\tau=1}^{s}c_{\tau-q}\{\sum_{h=1}^{\infty}\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]a_{h-q,\tau}\}$$

$$\leq\sum_{\tau=1}^{s}c_{\tau-q}\{q(D-G)[\delta(q+1)-1]\}$$

$$=q(D-G)[\delta(q+1)-1]\sum_{\tau=1}^{s}c_{\tau-q}=q(D-G)[\delta(q+1)-1].$$

A growth and distortion property for functions in the subclass $\mathcal{M}_q^{h,x}(\delta,G,D)$ is provided in the next two theorems.

Theorem (3.6): If $f\in\mathcal{M}_q^{h,x}(\delta,G,D)$ given by (1.1), then for $0<|z|=r<1$, we have

$$\frac{1}{r^q}\left(1-\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]}\right)\leq|f(z)|$$

$$\leq\frac{1}{r^q}\left(1+\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]}\right).$$

***Proof*:** Since $f(z)=\frac{1}{z^q}+\sum_{h=1}^{\infty}a_{h-q}z^{h-q}$,($a_{h-q}\geq 0,\ \ q\in\mathbb{N}$), then

$$|f(z)|=\left|\frac{1}{z^q}+\sum_{h=1}^{\infty}a_{h-q}z^{h-q}\right|$$

$$\leq\frac{1}{|z|^q}+\sum_{h=1}^{\infty}a_{h-q}|z|^{h-q}$$

$$\leq\frac{1}{r^q}\left(1+\sum_{h=1}^{\infty}a_{h-q}\right).\qquad(3.2)$$

From Corollary (3.2) $a_{h-q}\leq\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]},(h\geq 1)$, hence from (3.2) we obtain

$$|f(z)| \leq \frac{1}{r^q}\left(1+\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^x(1-D)(h-q)[\delta(h-q-1)+1]}\right),$$

similarly,

$$|f(z)| \geq \frac{1}{|z|^q}+\sum_{h=1}^{\infty} a_{h-q}|z|^{h-q}$$

$$\geq \frac{1}{r^q}\left(1+\sum_{h=1}^{\infty} a_{h-q}\right)$$

$$\geq \frac{1}{r^q}\left(1-\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^x(1-D)(h-q)[\delta(h-q-1)+1]}\right).$$

Theorem (3.7): If $f \in \mathcal{M}_q^{h,x}(\delta,G,D)$ given by (1.1), then for $0<|z|=r<1$, we have

$$\frac{q}{r^{q+1}}\left(1-r\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^x(1-D)(h-q)[\delta(h-q-1)+1]}\right) \leq |f'(z)|$$

$$\leq \frac{q}{r^{q+1}}\left(1+r\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^x(1-D)(h-q)[\delta(h-q-1)+1]}\right).$$

***Proof*:** Assume that $f \in \mathcal{M}_q^{h,x}(\delta,G,D)$, then

$$|f'(z)| \leq \left|\frac{q}{z^{q+1}}+\sum_{h=1}^{\infty} a_{h-q}(h-q)z^{h-q-1}\right|$$

$$\leq \frac{q}{|z|^{q+1}}+\sum_{h=1}^{\infty} a_{h-q}(h-q)|z|^{q\,h-q-1}$$

$$\leq \frac{q}{z^{q+1}}\left(1+r\sum_{h=1}^{\infty} a_{h-q}\right)$$

$$\leq \frac{q}{z^{q+1}}\left(1+r\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^x(1-D)(h-q)[\delta(h-q-1)+1]}\right).$$

Also, we have

$$|f'(z)| = \left|\frac{q}{z^{q+1}} - \sum_{h=1}^{\infty} a_{h-q}(h-q)z^{h-q-1}\right|$$

$$\geq \frac{q}{|z|^{q+1}} - \sum_{h=1}^{\infty} a_{h-q}(h-q)|z|^{q\,h-q-1}$$

$$\geq \frac{q}{z^{q+1}}(1 - r\sum_{h=1}^{\infty} a_{h-q})$$

$$\geq \frac{q}{z^{q+1}}(1 - r\frac{q(D-G)[\delta(q+1)-1]}{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]}.$$

Thus, the proof is complete.

The following theorems define the partial sum property and indicate the radii of starlikeness and convexity.

Theorem (3.8): If $f \in \mathcal{M}_q^{h,x}(\delta, G, D),$ Consequently, $f(z)$ is starlike and meromorphically univalent of order β in $|z| < k_1, (0 \leq \beta < q)$ and meromorphically univalent convex of order η in $|z| < k_2 (0 \leq \eta < 1)$ where

$$k_1 = ihf_h\left\{\frac{\left[\frac{1}{h+q+1}\right]^{x}(q-\beta)(1-D)(h-q)[\delta(h-q-1)+1]}{q(h+q-\beta)(D-G)[\delta(q+1)-1]}\right\}^{\frac{1}{h}}, (h \geq 1). \quad (3.3)$$

And

$$k_2 = ihf_h\left\{\frac{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q-1)+1]}{(h+q-\eta)(D-G)[\delta(q+1)-1]}\right\}^{\frac{1}{h}}, (h \geq 1). \quad (3.4)$$

***Proof*:** To prove $\left|q + \frac{zf'(z)}{f(z)}\right| \leq q - \beta,\ |z| < k_1$ for starlikeness

Since $\left|q + \frac{zf'(z)}{f(z)}\right| = \left|\frac{\sum_{h=1}^{\infty} h a_{h-q} z^h}{1+\sum_{h=1}^{\infty} a_{h-q} z^h}\right| \leq \frac{\sum_{h=1}^{\infty} h a_{h-q}|z|^h}{1-\sum_{h=1}^{\infty} a_{h-q}|z|^h} \leq q - \beta,\ |z| < k_1$

or $\sum_{h=1}^{\infty} h a_{h-q}|z|^h \leq (q-\beta) - (q-\beta)\sum_{h=1}^{\infty} a_{h-q}\,|z|^h$ or $\sum_{h=1}^{\infty} \frac{h+q-\beta}{q-\beta} a_{h-q}|z|^h \leq 1.$

From Corollary (3.2) $\frac{h+q-\beta}{q-\beta}|z|^h \leq \frac{\left[\frac{1}{h+q+1}\right]^x (1-D)(h-q)[\delta(h-q-1)+1]}{q(D-G)[\delta(q+1)-1]}$,

$$|z|^h \leq \frac{\left[\frac{1}{h+q+1}\right]^x (q-\beta)(1-D)(h-q)[\delta(h-q-1)+1]}{q(h+q-\eta)(D-G)[\delta(q+1)-1]}.$$

Thus, we obtain (3.3). For convexity, using Alexander's Theorem $f(z)$ is convex if and only if $zf'(z)$ is starlike we conclude (3.4) which is completes the proof.

Theorem (3.9): If $f \in \mathcal{M}_q^{h,x}(\delta, G, D)$ given by (1.1), $-1 \leq D < G \leq 1$ and $\mathcal{R}_1(z) = \frac{1}{z^q}, \mathcal{R}_s(z) = \frac{1}{z} + \sum_{h=1}^{s-1} a_{h-q} z^{h-q}, (s = 2,3,\dots)$.

Also suppose that $\sum_{h=1}^{\infty} y_{h-q} a_{h-q} \leq 1$ where

$$y_{h-q} = \frac{\left[\frac{1}{h+q+1}\right]^x (1-D)(h-q)[\delta(h-q-1)+1]}{q(D-G)[\delta(q+1)-1]}, \quad (3.5)$$

then

$$Re\left(\frac{f(z)}{\mathcal{R}_s(z)}\right) > 1 - \frac{1}{y_s}, \quad Re\left(\frac{\mathcal{R}_s(z)}{f(z)}\right) > \frac{y_s}{y_s+1}. \quad (3.6)$$

***Proof*:** Since $\sum_{h=1}^{\infty} y_{h-q} a_{h-q} \leq 1$ from Theorem (3.1) $f \in \mathcal{M}_q^{h,x}(\delta, G, D)$ and $y_{h-q+1} > y_{h-q} > 1, h \geq 1$, therefore

$$\sum_{h=1}^{s-1} a_{h-q} + y_s \sum_{h=s}^{\infty} a_{h-q} \leq \sum_{h=1}^{\infty} y_{h-q} a_{h-q} \leq 1. \quad (3.7)$$

By putting $\mathcal{H}_1(z) = y_s\left[\frac{f(z)}{\mathcal{R}_s(z)} - \left(1 - \frac{1}{y_s}\right)\right] = \frac{y_s \sum_{h=1}^{s-1} a_{h-q} z^{h-q}}{1+\sum_{h=1}^{s-1} a_{h-q} z^{h-q}} + 1$ and applying (3.7) we have

$$Re\left(\frac{y_s(z)-1}{\mathcal{H}_1(z)+1}\right) = \left|\frac{y_s(z)-1}{\mathcal{H}_1(z)+1}\right|$$

$$= \left|\frac{y_s \sum_{h=1}^{\infty} a_{h-q} z^{h-q}}{2 + y_s \sum_{h=1}^{\infty} a_{h-q} z^{h-q} + 2y_s \sum_{h=1}^{s-1} a_{h-q} z^{h-q}}\right|$$

$$\leq \frac{y_s \sum_{h=1}^{\infty} a_{h-q} z^{h-q}}{2 - y_s \sum_{h=1}^{\infty} a_{h-q} z^{h-q} - 2\sum_{h=1}^{s-1} a_{h-q} z^{h-q}} \leq 1.$$

A quick computation yields $Re(\mathcal{H}_1(z)) > 0$ and $Re\left(\frac{\mathcal{H}_1(z)}{y_s}\right) > 0$ which is equivalently to

$$Re\left(\frac{f(z)}{\mathcal{R}_s(z)} - \left(1 - \frac{1}{y_s}\right)\right) > 0$$

The first inequality in (3.6) is produced by this. Regarding the second inequality, we think

$$\mathcal{H}_1(z) = (y_s + 1)\left(\frac{\mathcal{R}_s(z)}{f(z)} - \frac{y_s}{y_s+1}\right) = 1 - \frac{(y_s+1)\sum_{h=1}^{\infty} a_{h-q} z^{h-q}}{1+\sum_{h=1}^{\infty} a_{h-q} z^{h-q}},$$

and by using (3.7)

$\left|\frac{\mathcal{H}_2(z)-1}{\mathcal{H}_2(z)+1}\right| \leq 1$. Thus $Re(\mathcal{H}_2(z)) > 0$ therefore $Re\left(\frac{\mathcal{H}_2(z)}{y_s+1}\right) > 0$ which is equivalently to

$$Re\left(\frac{\mathcal{R}_s(z)}{f(z)} - \frac{y_s}{y_s+1}\right) > 0.$$

Hence, the proof is complete.

Theorem (3.10): This theorem for neighborhoods results, Some ideas, such as neighborhood for analytic univalent functions, will be introduced in this theorem. For analytical univalent functions, Ruscheweyh [14], Raina and Srivastava [15], and Lashin [16] have the necessary neighborhood we define ϵ- neighborhood of $f \in \mathcal{M}_q^h$ given by (1.1) as

$$N_\epsilon(z) = \left\{g(z): g(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} c_{h-q} z^{h-q} \in \mathcal{M}_q^h, \lambda \leq \epsilon\right\}.$$

Where

$$\lambda = \sum_{h=1}^{\infty} \frac{\left[\frac{1}{h+q+1}\right]^x (1-D)(h-q)[\delta(h-q-1)+1]}{q(D-G)[\delta(q+1)-1]} \left|a_{h-q} - c_{h-q}\right|.$$

Now, if $f \in \mathcal{M}_q^{h,x}(\delta, G, D)$ given by (1.1) and f satisfies the condition

$$\frac{\vartheta z^{-q}+f(z)}{\vartheta+1} \in \mathcal{M}_q^{h,x}(\delta,G,D)(|\vartheta|<\epsilon,\epsilon>0,\vartheta\in\mathbb{C}),$$

then $N_\epsilon(f) \subset \mathcal{M}_q^{h,x}(\delta,G,D)$.

***Proof*:** by using (2.1), we have $f \in \mathcal{M}_q^{h,x}(\delta,G,D)$ if and only if

$$\frac{\delta z^{q+2}[\mathfrak{Z}_q^x f(z)]'' + z^{q+1}[\mathfrak{Z}_q^x f(z)]' - q[\delta(q+1)-1]}{D\left[\delta z^{q+2}[\mathfrak{Z}_q^x f(z)]'' + z^{q+1}[\mathfrak{Z}_q^x f(z)]'\right] - Gq[\delta(q+1)-1]} \neq \psi,$$

for any $|\psi| = 1, \psi \in \mathbb{C}$ or $\frac{(f*K)(z)}{z^{-q}} \neq 0, (z \in U^*),$ (3.8)

where $K(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} d_{h-q} z^{h-q}$. Such that

$$d_{h-q} = \frac{\left[\frac{1}{h+q+1}\right]^x (1-D\psi)(h-q)[\delta(h-q-1)+1]}{q\psi(D-G)[\delta(q+1)-1]}, \tag{3.9}$$

$$|d_{h-q}| = \left|\frac{\left[\frac{1}{h+q+1}\right]^x (1-D\psi)(h-q)[\delta(h-q-1)+1]}{q\psi(D-G)[\delta(q+1)-1]}\right|$$

$$\leq \frac{\left[\frac{1}{h+q+1}\right]^x (q-\beta)(1-D)(h-q)[\delta(h-q-1)+1]}{q(D-G)[\delta(q+1)-1]},$$

since

$$\frac{\vartheta z^{-q}+f(z)}{\vartheta+1} \in \mathcal{M}_q^{h,x}(\delta,G,D)(|\vartheta|<\epsilon,\epsilon>0,\vartheta\in\mathbb{C}),$$

we get

$$\frac{\left(\frac{\vartheta z^{-q}+f(z)}{\vartheta+1} * K\right)(z)}{z^{-q}} \neq 0, (z \in U^*). \tag{3.10}$$

If we assume that $\left|\frac{(f*K)(z)}{z^{-q}}\right| < \epsilon$, then by (3.10), we have

$$\left|\frac{(f*K)(z)}{(\vartheta+1)z^{-q}} + \frac{\vartheta}{(\vartheta+1)}\right| \geq \frac{|\vartheta|-1}{|\vartheta+1|}\left|\frac{(f*K)(z)}{(z^{-q}}\right| > \frac{|\vartheta|-\epsilon}{|\vartheta+1|} \geq 0,$$

but this is contradiction with $|\vartheta| < \epsilon$, therefore $\left|\frac{(f*K)(z)}{z^{-q}}\right| < \epsilon$.

Now, if we suppose that $g(z) = \frac{1}{z^q} + \sum_{h=1}^{\infty} c_{h-q} z^{h-q} \in N_\epsilon(f)$, then

$$\left|\frac{(f-g)(z)*K)(z)}{z^{-q}}\right| = \left|\sum_{h=1}^{\infty}(a_{h-q} - c_{h-q})d_{h-q} z^{h-q}\right|$$

$$\leq \sum_{h=1}^{\infty}\left|a_{h-q} - c_{h-q}\right|\left|d_{h-q}\right|\left|z^{h-q}\right|$$

$$\leq \left|z^{h-q}\right| \sum_{h=1}^{\infty} \frac{\left[\frac{1}{h+q+1}\right]^{x}(1-D)(h-q)[\delta(h-q+1)+1]}{q(D-G)[\delta(q+1)-1]}\left|a_{h-q} - c_{h-q}\right| \leq \epsilon.$$

Hence, we get

$$\frac{(g*K)(z)}{z^{-q}} \neq 0, (z \in U^*),$$

for any $\psi \in \mathbb{C}, |\psi| = 1$, therefore $g(z) \in \mathcal{M}_q^{h,x}(\delta, G, D)$ which implies that

$$N_\epsilon(f) \subset \mathcal{M}_q^{h,x}(\delta, G, D).$$

Conclusion and Future Studies

In this article, we introduced a subclass of meromorphic multivalent functions Defined by integral operator that are important for mathematical finance. As a outcome, we deduced some geometric properties of coefficient estimates, extreme points, convex linear combination, radii of starlikeness and convexity. Furthermore, we investigated the neighborhood of the presented classes.

Finally, we suggest in the future studies the coefficient problems continue to include new classes of other functions and new generalized derivative operators. Also, it is possible to study new properties other than those mentioned in this article.

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