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# Semi-QUASI HAMSHER MODULES

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**Abstract:**

This study presents a semi-quasi Hamsher module that each non-zero Artinian submodule has a semi-maximal submodule, which is a generalized of Hamsher module. And semi-quasi Loewy module that each non-zero Noetherian submodule has a semi-maximal submodule, which is a generalized of Loewy module. This article introduces some properties of semi-quasi Hamsher and semi-quasi Loewy modules .

Keywords: semi-quasi Hamsher module, semi-quasi Loewy module, semi-maximal submodule and semi-socle submodule .

**Introduction:**

Throughout rings and modules are unitary. We use the terminology and notations of Anderson and Fuller[1]. Faith [2] defined a module X is Hamsher if each non-zero submodule of X has a maximal submodule. In[3] we see that X has finite length if and only if X is Hamsher and Artinian. Weimin[4] generalized to quasi-Hamsher module X if every non-zero Artinian submodule of X has a maximal submodule. A ring R is said to be right maximal if each non-zero right R-module has a maximal submodule[2].This class of rings includes right perfect rings. In this paper, we characterize semi-quasi-Hamsher module(for short; S.Q.Ham.Mod) if each non-zero Artinian submodule has a semi-maximal submodule(for short; s-max.sub).

**1-S.Q.Ham.Mod:** The class of S.Q.Ham.Mod is closed under submodules, also closed under extensions, direct products, and direct sums as we see in the following propositions .

Proposition(1.1)*Let* $0 \to X_1 \xrightarrow{f} X \xrightarrow{g} X_2 \to 0$ be an exact sequence of modules. If $X_1$ and $X_2$ are S.Q.Ham.Mod, then so is X.

Proof: Let L≠0 be an Artinian submodule of X. If g(L) # 0, being an Artinian submodule of the S.Q.Ham.Mod $X_2$, g(L) has a s-max.sub N. Then $L \cap g^{-1}(N)$ is a s-max.sub of L. If $g(L) = 0, L \subseteq \ker(g) = Im(f) \cong X_1$ , so L has a s-max.sub since X is S.Q.Ham.Mod .

Proposition (1.2) Let {Xi}S.Q.Ham.Mod be a family of modules, then the following statements are equivalent :

1-Each Mi is S.Q.Ham.Mod

2-$\prod_{i\epsilon I} X_i$ *is* S.Q.Ham.Mod

3-$\bigoplus_{i\in I} X_i$ *is* S.Q.Ham.Mod

Proof: (1) ⇒ (2) Let L≠0 be an Artinian submodule of $\prod_{i\epsilon I} X_i$ , and let $f_i$ : $\prod_{i\epsilon I} X_i$ →$X_i$ be a canonical projections. We have $X_i$ such that $f_i(L) \neq 0$. Then $f_i$ (L) is an Artinian submodule of S.Q.Ham.Mod Xi so $f_i(L)$ has a s-max.sub N. Thus $L\cap f^{-1}(N)$ is a s-max.sub of L[6], therefore $\prod_{i\epsilon I} X_i$ is S.Q.Ham.Mod .

(2) ⇒ (3) ⇒ (1) These are obvious because the class of S.Q.Ham.Mod is closed under submodules. Cai and Xue[5] called a module X is strongly Artinian if each of its proper submodule has finite length. It is easy to see that a non-zero strongly Artinian module has finite length if and only if it has a maximal submodule if and only if it is finitely generated. And since every maximal submodule is a semi-maximal submodule, thus we can see that a module X is semi-strongly Artinian if every of its proper submodule has finite length[6]. So we can say that a non-zero semi-strongly Artinian module has finite length if and only if it has a semi-maximal submodule if and only if it is finitely generated . Proposition(1.3) the following statements are equivalent :

1- X is S.Q.Ham.Mod;
2-Each Artinian submodule of X has finite length
3-Each Artinian submodule of X is finitely generated
4-Each semi-strongly Artinian submodule of X is finitely generated
5-Each non-zero semi-strongly Artinian submodule of X has a semi-maximal submodule, so it has finite length .

Proof: (1) ⇒ **(2)**Let L be a non-zero Artinian submodule of X. Since each submodule of L is still Artinian, L is an Artinian Hamsher module, which has finite length.

(2) ⇒ **(3)** ⇒ **(4)** ⇔ **(5)** and **(3)** ⇒ **(1)** Thes are obvious .

(3) ⇒ (2) If L is an Artinian submodule of X and L has infinite length, then the non-empty family { N ⊆ L | N has an infinite length } has a minimal member, say N. It is easy to see that N is strongly Artinian and N has infinite length .

As a generalization of maximal module the module X is semi-maximal if it is semi-simple[7], and [8] defined quasi-maximal module if Rad($ann_R$ X) is semi-maximal ideal of a ring R. Also a ring R is said to be right semi-maximal if each non-zero right R-module has semi-maximal submodule[9], and we call a ring R is right semi quasi maximal if every right R-module is semi quasi Hamsher. The next characterizations of right semi quasi maximal rings follow immediately from the above proposition.

Theorem(1.4) The following statements are equivalent :
1-R is right semi quasi maximal ring
2-Every non-zero strongly Artinian right R-module has a semi-maximal submodule
3-Every (strongly) Artinian right R-module has finite length;
4-Every (strongly) Artinian right R-module is finitely generated.

Camillo and Xue [3] called a ring R right quasi-perfect if every Artinian right R-module has a projective cover. Using Th.(1.4) and [3], we see that a ring R is right quasi perfect if and only if it is semi perfect and right quasi semi-maximal[3]

Proposition(1.5) If R is commutative semi perfect ring with nil J(R), then R is semi-quasi maximal ring .

A ring R is right maximal if and only if R/J(R) is right semi-maximal and J(R) is right T-nilpotent[5]. The ring R is a local commutative ring with nil J(R) which is not T-nilpotent. Hence R is not maximal[3], but R is semi-quasi maximal (Prop.1.5). Therefore there is a semi-quasi-Hamsher R-module which is not Hamsher. We conclude that semi-quasi-Hamsher modules and right semi-quasi-maximal rings are proper generalizations of Hamsher modules and right semi-maximal rings, respectively.

Example(1.6) Let V be a division ring. Let R be the ring of all countable infinite upper triangular matrixes over V with constant on the main diagonal and having non-zero entries in only finitely many rows above the main diagonal. Then R is a local right perfect ring which is not left perfect. Miller and Turnidge [10] constructed and Artinian left R-module X which is not Noetherian. Hence R is not left semi-quasi maximal. This shows that the notion of semi-quasi maximal rings is not left-right symmetric.

In view of the above example and prop.(1.5), we mention the following result .

Proposition(1.7) Let R be a semi perfect ring with nil J(R). If J(R) is of bounded index n, i.e.( $j^n = 0$) for each j ϵ J (R), then R is semi-quasi-maximal or semi-quasi-perfect.

Modifying the proof of [2] we have an analogous result.

Theorem(1.8): The following statements are equivalent

1-R is right quasi-maximal ring

2-The category Mod-R has a cogenerate G which is S.Q.Ham.Mod

3-The injective envelope E(X) of X is S.Q.Ham.Mod for each simple right module X.

Proof: (1) ⇒ (2) This is obvious.

(2) ⇒ (3) Since G is a cogenerator there is a monomorphism E(X) G for each simple right R-module X. Hence E(X) must be S.Q.Ham.Mod, since G is.

(3) ⇒ (1) Let X range over all simple right R-modules. Then ⨁E(X) is a co-generator of Mod-R and ⨁E(X) is S.Q.Ham.Mod by prop.(1.2) Let A be a non-zero Artinian right R-module. We have a non-zero homo. f:L→ ⨁E(X). Since f(L) is a non-zero Artinian submodule of E(X), which is S.Q.Ham.Mod, f(L) has a semi-maximal submodule of N. Then $f^{-1}(N)$ is a semi-maximal submodule of L .

**2-Semi-Quasi Loewy Modules :** [12]Recall that a module M is called Loewy if every non-zero factor module of M has non-zero socle. And a module M is called quasi-Loewy if every non-zero Noetherian module of M has non-zero socle[4]. A module M has finite length if and only if M is Loewy and Noetherian [1]. A module X is semi-local if $\frac{X}{Rad(X)}$ is semi-simple[13]. In this section we interduce a concept that a module X is semi-quasi Loewy module(for short; S-Q Loy. Mod.) if every non-zero Noetherian module of X has non-zero semi-socle. The next two propositions show that the class of S-Q Loy. Mod. is closed under extensions and direct sums.

Proposition(2.1) $Let\ 0 \to X_1 \xrightarrow{f} X \xrightarrow{g} X_2 \to 0$ be an exact sequence of modules. If both $X_1$ and $X_2$ are S-Q Loy. Mods., then X is S-Q Loy. Mod.

Proof: $Let\ \frac{X}{L} \neq 0\ be\ a\ factor\ module\ of\ X.$

$$We\ have\ an\ exact\ sequence\ \ 0 \to \frac{X_1}{L_1} \to \frac{X}{L} \to \frac{X_2}{L_2} \to 0$$

$$If\ \frac{X_1}{L_1} \neq 0\ and\ soc\left(\frac{X_1}{L_1}\right) \neq 0, then\ soc\left(\frac{X}{L}\right) \neq 0.$$

$$If\ \frac{X_1}{L_1} = 0, \frac{X_2}{L_2} \cong \frac{X}{L} \neq 0. Then\ soc\left(\frac{X_2}{L_2}\right) \neq 0\ and\ soc\left(\frac{X}{L}\right) \neq 0.$$

Proposition(2.2) Let $\{X_i\}_{i\epsilon I}$ be a family of modules. W is S-Q Loy. Mod. if and only if every $X_i$ is S-Q Loy. Mod..

Proof: The class of S-Q Loy. Mod. is closed under factor modules .
Conversily; $let\ f_i: X_i \to \bigoplus_{i\epsilon I} X_i$ be a canonical injection.

$If\ \frac{\bigoplus_{i\epsilon I} X_i}{L}$ is a non-zero (Noetherian) factor module of $\bigoplus_{i\epsilon I} X_i\ , then\ there\ is\ i\epsilon I\ such\ that\ 0 \neq g_i: X_i \to \frac{\bigoplus_{i\epsilon I} X_i}{L}$

$where\ g: \bigoplus_{i\epsilon I} X_i \to \frac{\bigoplus_{i\epsilon I} X_i}{L}$ is the natural epimorphism.

Since $Im(g_{ji}) \neq 0$ which is isomorphic to a (Noetherian) factor module of $X_i\ , we\ have\ 0\ \#\ soc(Im(gji)) \subseteq soc\left(\frac{\bigoplus_{i\epsilon I} X_i}{L}\right).$

$If\ R = \prod_{i=1}^{\infty} P_i$ is an infinite product of the fields $P_i$
then R is not a Loewy module [8]. Since every $P_i$ is a Loewy module, this shows that the class of Loewy modules is not closed under direct products. We do not know if the class of S-Q Loy. Mod. is closed under direct products.
A module is called strongly Noetherian if each of its proper factor module has finite length[14],[15]. It is easy to see that a non-zero strongly Noetherian module has finite length if and only if it has non-zero semi-socle if and only if it is finitely cogenerated[6] .
Proposition (2.3) The following statements are equivalent:

1-X is S-Q Loy. Mod.

2-Each Noetherian factor module of X has finite length
3-Each Noetherian factor module of X is finitely cogenerated
4-Each strongly Noetherian factor module of X is finitely cogenerated
5-Each non-zero strongly Noetherian factor module of X has non-zero semi-soc and finite length

Proof: (1) ⇒(2) Let $\frac{X}{L} \neq 0$ be a Noetherian factor module of X. Since each factor module of $\frac{X}{L}$ is still Noetherian, thus $\frac{X}{L}$ has finite length .

(2) ⇒(3) ⇒(4) ⇔ (5) and (3) ⇒(1) These are obvious .

(5) ⇒(2) *If* $\frac{X}{L}$ is a Noetherian factor module of X and $\frac{X}{L}$ has infinite length, then the non-empty family $\{L \subseteq L` \subseteq X \mid \frac{X}{L}\ has\ infinte\ length\}$ .

*has a maximal member say L`.Thus* $\frac{X}{L`}$ *is strongly Noetherian and*

has infinite length .

A ring R is called right S-Q Loy. if every right R-module is S-Q Loy. . The next characterizations of right S-Q Loy. rings follow immediately from the above proposition.

Theorem(2.4) The following statements are equivalent :
1-R is right S-Q Loy. ring

2-Each non-zero (strongly) Noetherian right R-module has non-zero semi-socle
3-Each (strongly) Noetherian right R-module has finite length

4-Each (strongly) Noetherian right R-module is finitely co-generated

It follows from Th.(1.4) and Th.(2.4) that the rings studied by Tanabe [11] are precisely left semi-quasi maximal and left S-Q Loy. rings. An analogous result of Th.(1.8) is the following

Theorem (2.5) A ring R is right S-Q Loy. if and only if Mod-R has a generator C which is S-Q Loy.

Proof: If X is a Noetherian right R − module $X \cong \frac{C^n}{L}$ .

$C^n$ is S-Q Loy. prop.(2.2), so $\frac{C^n}{L}$ has finite length prop.(2.3). Hence R is right S-Q Loy. th.(2.4) .

The convers is clear

The next proposition gives a class of commutative S-Q Loy. rings.

Proposition (2.6) If R is a commutative semi-perfect ring with nil J(R) then R is S-Q Loy. Mod. ring .

Proof. By Th.(2.5), it suffices to show that R is a S-Q Loy. Mod. . Let A be an ideal of R such that $\frac{R}{A}$ is a Noetherian R-module .

Then the commutative semi − perfect Noetherian ring $\frac{R}{A}$ has nil $J\left(\frac{R}{A}\right)$.

Hence $\frac{R}{A}$ is an Artinian ring. Then $\frac{R}{A}$ has finite length as an R-module .

R is right Loewy ring if every right R-module is Loewy[4], its mean every non-zero right R-module has non-zero socle, equivalently, the right R-module $R_R$ is Loewy. Every left perfect ring is right Loewy. In [16] R is right Loewy if and only if $\frac{R}{J(R)}$ R is right Loewy, and J (R) is left X-nilpotent. The ring R in [3] is a local commutative ring with nil J (R) which is not X-nilpotent. Hence R is not Loewy, prop.(2-6) but R is semi-quasi. Therefore there is a S-Q Loy. Mod. which is not Loewy. We conclude that S-Q Loy. Mod. and S-Q Loy. rings are proper generalizations of Loewy modules and right Loewy rings, respectively.
Let R be the ring in ex.(1.6) Then R is a local right perfect ring which is not left perfect[17]. Miller and Turnidge [6] constructed a Noetherian right module X which is not Artinian. Hence R is not right S-Q Loy. Mod. .This shows that the notion of S-Q Loy. rings is not left-right symmetric. In view of this fact and prop.(2.6), we state the next result, which follows from [11] .

Proposition (2.7) Let R be a semiperfect ring with nil J(R). If J (R) is of bounded index n then R is (two-sided) S-Q Loy. ring .

Since a commutative regular ring need not be Loewy (see $R = \prod_{i=1}^{\infty} P_i$ preceding prop. 2.3),

Proposition (2.8) Every strongly regular ring R is a (two-sided) S-Q Loy. ring .

$$Proof: let \sum_{i=1}^{n} x_i R \ be\ a\ Noetherian\ right$$

$$R - module. It\ suffices\ to\ show\ X$$

has finite length, we have

$$x_i R \cong \frac{R}{A} \text{ for some ideal A of R, since } \frac{R}{A} \ is\ a\ right$$

Noetherian regular ring it is semi − simple,

and $\frac{R}{A} \cong x_i R$ has finite length .

**3-Semi-Quasi Hamsher Rings(S-Q Ham. Rings)**

Morita duality. A bimodule $_GI_R$ defines a Morita duality if $_GI_R$ is faithfully balanced and both $I_R$ and $_GI$ and $I_R$ are injective co-generators. In this case, both R and G are semi-perfect rings. In [18] we can see a presentation of Morita duality, by using properties of Morita duality .

Proposition(3.1) Let $_GI_R$ define a Morita duality. If $X_R$ is a I-reflective right module then

1- $X_R$ is S-Q Low. Mods if and only if the left G-module $_G\text{Horn}_I(X_R , {_GI_R})$ is S-Q Low. Rings .
2- $X_R$ is S-Q Ham. Mods if and only if the left G-module $_G\text{Horn}_I(X_R , {_GI_R})$ is S-Q Ham. Rings .

Theorem(3.2)If $_GI_R$ defines a Morita duality, then the following statements are equivalent:

1- R is right semi-quasi maximal
2- G is left semi-quasi maximal

3- R is right semi-quasi Loewy

4- G is left semi-quasi Loewy.

**Discussion and conclusion:** The aim of this manuscript is to introduced a new generalized of Hamsher module which is semi-quasi Hamsher module that each non-zero Artinian submodule has a semi-maximal submodule. This class of module is closed under extension, direct product and direct sum. Furthermore; we introduce a new generalized of Loewy module which is semi-quasi Loewy module that each non-zero

Noetherian submodule has a semi-socle submodule. This class of module is closed under extension, direct product and direct sum .

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