On converse theorems of trigonometric approximation in weighted spaces

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Abstract:

Many researchers in the field of functional analysis have proved the direct theorems for approximating functions in several known spaces. As for us in this work, we will prove inverse theorems for approximating unbounded functions using trigonometric polynomial in weighted spaces via modulus of smoothness functions.

Keywords: Maximal operator, Weighted spaces, weighted fractional moduli of smoothness, converse theorem

1.Introduction: There is an increasing interest in the spaces of functions with, bounded properties in Lebesgue spaces, whose is variable exponent , and there weighted spaces are based on homogeneous spaces .We also need the extinction theorems to determine these limits in these spaces for the various factors of harmonic analysis ,the trigonometric application has inverse estimates that determine the membership of a function in some of the smoothness classes(like Lipschitz class) , and it is known in terms of the approximation rate

$$\varphi_s\left(g,\frac{1}{m}\right) \le \frac{d_1}{m^s}\{\sum_{n=0}^{m}(n+1)^{s-1}\,\mathcal{E}_n\,(g)_t\} \ldots\ldots\ldots \quad (1.1)$$

The trigonometric approximation applies to Lebesgue spaces

$\mathcal{L}^t(\mathcal{F}), 1 \le p < \infty, or\, \mathcal{B}(\mathcal{F})$, (the continuous function on $\mathcal{F}$) for $t = \infty$, $(t = \infty, p < \infty)$ since $\mathcal{F} := [0, 2\pi), g \in \mathcal{L}^t(\mathcal{F}), 1 \le p < \infty$,

$s, m \in N := \{1, 2, 3, \dots\}$, $\mathcal{F}_f\, g(\circ) := g(\circ + f)$ *by translation operator*,
$\varphi_s(g, Q)_{p,\mu} := \sup\left\{\left\|\left(\mathcal{F}_f\right)^s g\right\|_{p,\mu} : 0 < f \le Q\right\}$
the s th moduli of smoothness of the function g , I is identity operator , $\mathcal{F}_m$ .There is a class of trigonometric polynomials whose degree is no more than m,

$$\mathcal{E}_m(g)_{p,\mu} := \inf\left\{\| g - \mathcal{F} \|_{p,\mu} : \mathcal{F} \in \mathcal{F}_m\right\}$$

where d_1 a constant and it just depends on s , t .After that , various applications of (1 .1) and generalizations were reached . In 1958 Timan proved this development (1 .1)also applies to

If $1 < p < \infty, g \in \mathcal{L}^t(\mathcal{F}), m, s \in N, p = \min\{2, t\}$ *then*

$$\varphi_s\left(g, \frac{1}{m}\right) \le \frac{d_2}{m^s}\left\{\sum_{n=1}^{m} n^{s\,p-1}\, \mathcal{E}_{n-1}^{p}(g)_{p,\mu}\right\}^{\frac{1}{p}} \qquad (1.2)$$

where d_2 a constant and it just depends on s , t

We noticed that the value $min\{2, t\}$ in (1.2) it is the optimal value , we notice that there are similar problems within the spaces of weighted functions , for example,Lebesgue spaces $L_{p,\mu}(X)$, weighted variable exponent spaces .Now we need different parameters for smoothness ,note the definition below

Let $\varphi \in B_{t,}, 1 < t < \infty, g \in L_{p,\mu}(X), s, m \in N$

Also let

$$\delta_f\, g(y) := \frac{1}{2f}\int_{y-f}^{y+f} g(x)dx \ \ for\ f \in R\ and\ y \in \mathcal{F}.$$

We now need to determine the modulus

$$\jmath_s(g, Q)_{p,\mu} := \sup_{0 \le f_j \le Q}\left\|\prod_{j=1}^{s}\left(I - \delta_{f_j}\right) g\right\|_{p,\mu}, Q \ge 0$$

Now

$$j_s\left(g,\frac{1}{m}\right)_{p,\mu} \le \frac{d_3}{m^{2s}}\left\{\mathcal{E}_0(g)_{p,\mu} + \sum_{n=1}^{m} n^{2s-1}\ \mathcal{E}_n\ (g)_{p,\mu}\right\} \ldots\ldots\ldots\ldots (1.3)$$

where d_3 a constant and it just depends on s , t

Now $1 < p < \infty$, $\varphi \in B_t$, $g \in L_{p,\mu}(X)$, $s, m \in N$, *where* d_4 *is* a positive constant and it just depends on s , t such that

$$j_s\left(g,\frac{1}{m}\right)_{p,\mu} \le \frac{d_4}{m^{2s}}\left\{\sum_{n=1}^{m} n^{2ps-1}\ \mathcal{E}_{n-1}^{p}\ (g)_{p,\mu}\right\}^{\frac{1}{p}} \ldots\ldots\ldots\ldots (1.4)$$

$\forall s \in \mathbb{R}^+$, now we will use the weighted fractional smoothness coefficients (1.4) ,the variable exponent is weighted with respect to $L_{p,\mu}(X)$ is (1.4) with $s \in \mathbb{R}^+$. In our work , we now prove the right side of (1.4) $2s$ and replace it with s. Variable exponent $L_{p,\mu}(X)$. For the first time , the un weighted fractional coefficients of smoothness were evaluated in spaces by Taberski and Butzer in 1977. Now we will use some definitions . Suppose T be a class of Lebesgue fraction that are measurable

Then

$$t : \mathcal{F} \longrightarrow (1,\infty) \textit{ such that } 1 < t_* := essinfp_{\ y\in\mathcal{F}}(y) \le t^* := essinfp_{\ y\in\mathcal{F}}(y) < \infty$$

.

We will write the definition of the conjugate exponent of $t(y)$ as

$$\acute{t}(y) := t(y)/t(y) - 1$$

The class $\mathcal{L}_{2\pi}^{t(.)}$ we define it of 2π periodic measurable functions $g : \mathcal{F} \longrightarrow \mathbb{C}$ then

$$\int_{\mathcal{F}} |g(y)|^{t(y)}\ dy < \infty$$

$\mathbb{C}$ is the complex plane and $t \in T$.

The $\mathcal{L}_{2\pi}^{t(\cdot)}$ it is a class of a Banach space with the norm

$$\|g\|_{p,\mu} := \inf\left\{b > 0 : \int_{\mathcal{F}} \left|\frac{g(y)}{b}\right|^{t(y)} dy \leq 1\right\}.$$

The functions $\varphi : \mathcal{F} \longrightarrow [0,\infty]$ is called a weight if it is proven to be measurable almost everywhere positive , the functions of Lebesgue measurable $g : \mathcal{F} \rightarrow \mathbb{C}$ hold $\varphi g \in \mathcal{L}_{2\pi}^{t(.)}$ $, \mathcal{L}_{\varphi}^{t(\cdot)}$

and we called weighted Lebesgue spaces with variable exponent and is Banach space with norm

$$\|g\|_{p,\mu} := \|\varphi g\|_{p,\mu}$$

And 2π periodic weight φ then we can denote by $\mathcal{L}_{\varphi}^{t}$ the weighted Lebesgue spaces 2π periodic measurable functions $g : \mathcal{F} \rightarrow \mathbb{C}$ then

$$g\,\varphi^{\frac{1}{t}} \in \mathcal{L}^{t}(\mathcal{F}) the\ set\ \|g\|_{p,\mu} := \left\|g\,\varphi^{\frac{1}{t}}\right\|_{p,\mu} for\ g \in \mathcal{L}_{\varphi}^{t}$$

We take $t \in T$, the following condition [1] is fulfilled by the weight class

$$\|\varphi^{t(x)}\|_{p,\mu} := \sup_{r\in\mathcal{R}} \frac{1}{\|R\|_{p,\mu}^{\ \ TR}} \|\varphi^{t(x)}\|_{p,\mu} \left\|\frac{1}{\varphi^{t(x)}}\right\|_{p,\mu} < \infty$$

We will use $B_{t(\ .\ )}$ $to\ refer\ to\ it.\ Since$

$$T_R := \left(\frac{1}{\|R\|_{p,\mu}} \int_R \frac{1}{t(y)}\, dy\right)^{-1} also\ \mathcal{R}\ is\ the\ class\ of\ all\ intervals\ in\ \mathcal{F}$$

There is a condition and there is a positive constant d_5 , so it is said to $T(y)$ it satisfies local log-Holder continuity

$$\|T(y_1) - T(y_2)\|_{p,\mu} \le \frac{d_5}{\log\left(\frac{e+1}{\|y_1 + y_2\|_{p,\mu}}\right)} \; \forall y_1, y_2 \in \mathcal{F} \dots\dots\dots\dots\dots (1.5)$$

The class $t \in T$ satisfying (1.5), now we will refer to it as T^{log}

Suppose $g \in \mathcal{L}_{\varphi}^{t(.)}$ and

$$\mathcal{H}_f\, g(y) := \frac{1}{f} \int_{y-f/2}^{y+f/2} g(x)\, dx, \quad y \in \mathcal{F}$$

The Hardy Littlewood maximum operator $\mathcal{U}$ is bounded by $\mathcal{L}_{\varphi}^{t(.)}$ if and only if $\varphi \in B_{t(.)}$ it has been proven in condition [1] and $t \in T^{log}$ Steklo$v$'s mean operator , if $t \in T^{log}$ and $\varphi \in B_{t(.)}, then\, B_f$ is bounded in $\mathcal{L}_{\varphi}^{t(.)}$. After using numbers and facts $y, f \in \mathcal{F}, 0 \le s$, we will determine through binomial expansion that

$$\mathcal{S}_f^s\, g(y) = (\mathcal{H}_f - J)^s\, g(y)$$

$$= \sum_{c=0}^{\infty} (-1)^c \binom{s}{c} \frac{1}{f^c} \int_{-f/2}^{f/2} \dots \int_{-f/2}^{f/2} g(y + h_1 + h_c)\, dh_1 \dots dh_c,$$

Where

$$g \in \mathcal{L}_{\varphi}^{t(.)}, \qquad \binom{s}{c} := \frac{s(s-1)\dots(s-f+1)}{c!}\; for\; c > 0,$$

$$\binom{s}{1} := s\; and\; \binom{s}{0} := 1, \qquad \sum_{c=0}^{\infty} \left|\binom{s}{c}\right| < \infty$$

If $t \in T^{log}, \varphi \in B_{t(.)}\; and\; g \in \mathcal{L}_{\varphi}^{t(.)}, there\; is\; apositive\; constant\; d_6$

Just depends on $s\; and\; t$ then

$$\left\|\mathcal{S}_f^s\, g\right\|_{p,\mu} \le d_6\; \|g\|_{p,\mu,} < \infty \dots\dots\dots\dots (1.6)$$

For $0 \leq s$ we can now determine the fractional moduli of smoothness below s for

$t \in T^{log}, \varphi \in B_{t(.)}$ and $g \in \mathcal{L}_{\varphi}^{t(.)}$

$$j_s(g, Q)_{p,\mu} := \sup_{0 < f_j, x \leq Q} \left\| \prod_{j=1}^{[s]} \left(J - \mathcal{H}_{f_j} \right) S_x^{s-[s]} g \right\|_{p,\mu,} , Q \geq 0,$$

Where

$$j_0(g, Q)_{p,\mu} := \|g\|_{p,\mu}; \prod_{j=1}^{0} \left(J - \mathcal{H}_{f_j} \right) S_x^s g := S_x^s g$$

Such that $0 < s < 1$ and $[s]$ it is the $integer$ $part$ of the $real$ $number$ s

We have that $t \in T^{log}, \varphi \in B_{t(.)}$ and $g \in \mathcal{L}_{\varphi}^{t(.)}$ by (1.6)

$there$ is apositive constant d_7 Just depends on s and t then

$j_s(g, Q)_{p,\mu} \leq d_7 \|g\|_{p,\mu}$.

If $t \in T^{log}, \varphi \in B_{t(.)}$, $then$ $\varphi^{t(y)} \in \mathcal{L}^1(\mathcal{F})$. The set of trigonometric polynomials in $\mathcal{L}_{\varphi}^{t(.)}$. On another side $t \in T^{log}$ and $\varphi \in B_{t(.)}$,

$then$ $\mathcal{L}_{\varphi}^{t(.)} \subset \mathcal{L}^1(\mathcal{F})$. For given $g \in \mathcal{L}^1(\mathcal{F})$,

suppose $g(y) \sim \frac{b_0(g)}{2} + \sum_{c=1}^{\infty} \left(b_c(g) \cos c\, y + r_c(g) \sin c\, y \right)$

$$= \sum_{c=-\infty}^{\infty} d_c(g) e^{j\,c\,y} \dots\dots\dots\dots\dots (1.7)$$

The Fourier series of g with $d_c(g) = \left(\frac{1}{2}\right) \left(b_c(g) - j\, r_c(g) \right)$, now we set

$\mathcal{L}_0^1(\mathcal{F}) := \{ g \in \mathcal{L}^1(\mathcal{F}) : d_0(g) = 0$ for the series in $(1.7)\}$.

Suppose $\beta \in \mathbb{R}^{+}$.We specify fractional derivative of a function $g \in \mathcal{L}_0^1(\mathcal{F})$ as

$$g^{(\beta)}(y) := \sum_{c=-\infty}^{\infty} d_c(g)\left((j\,c)_{p,\mu}\right)^{\beta} e^{j\,c\,y}$$

Provided that the right side is where $\left((j\,c)_{p,\mu}\right)^{\beta} := \|c\|_{p,\mu}\, e^{(1/2)\pi\, j\, \beta\, signk}$ as principal value . Now we talk about the function $g \in \mathcal{L}_{\varphi}^{t(.)}$ it has a fractional derivative of rank $\beta \in \mathbb{R}^{+}$ if *there is a* function $i \in \mathcal{L}_{\varphi}^{t(.)}$ so that it satisfy Fourier coefficients $d_c(\mathrm{i}) = d_c(\mathrm{g})\left((j\,c)_{p,\mu}\right)^{\beta}$ we will write in this case $g^{(\beta)} = \mathrm{i}$.

Suppose $K_{t(.),\varphi}^{\beta}$,t $\in$ T , β > 0 be the class of functions $g \in \mathcal{L}_{\varphi}^{t(.)}$ in which

$g^{(\beta)} \in \mathcal{L}_{\varphi}^{t(.)}$. $K_{t(.),\varphi}^{\beta}$

Becomes a Banach space with the norm

$$\|g\|_{p,\mu} := \|g\|_{p,\mu} + \left\|g^{(\beta)}\right\|_{p,\mu} .$$

The set $\mathcal{E}_m(g)_{p,\mu} := \inf\left\{\|g - \mathcal{F}\|_{p,\mu} : \mathcal{F} \in \mathcal{M}\right\}$ *for* $g \in \mathcal{L}_{\varphi}^{t(.)}$.

In 2014 H¨ast¨o and L. Diening, Muckenhoupt weights in variable exponent spaces [1] , studied variable exponent spaces was the hope that many classical results from Lebesgue space theory could be generalized to this setting, but not to general Musielak–Orlicz spaces , in 2003 , Kokilashvili and S.G. Samko, Singular integrals weighted Lebesgue spaces with variable exponent [2] , The mapping properties of Cauchy singular integrals defined on the Lyapunov curve and on curves of bounded rotation are also investigated, in 2007 , Kokilashvili and Yildirir, On the approximation in weighted Lebesgue [3] , the approximation problems for periodic functions are investigated in weighted Lebesgue spaces with the Muckenhoupt weights. For this case we obtain inverse type inequality for the derivatives of 2π periodic function in terms of generalized modulus of continuity. The introduction of such structural characteristic of functions was caused by the failure of continuity of shift operator in weighted spaces. In unweighted Lebesgue spaces the inequalities for classical modulus of continuity and the best approximations of derivatives , in 2007, Kokilashvili and Yildirir, On the approximation in weighted Lebesgue spaces [4] , we deal with the estimation of the best approximation and generalized modulus of continuity of derivatives of periodic functions in weighted reflexive Lebesgue spaces, in 2008 , Akgun and Israfilov, Approximation and moduli of fractional orders in Smirnov [5] , we investigate the approximation problems in the Smirnov-Orlicz spaces in terms of the fractional modulus of smoothness. We prove the direct and inverse theorems in these spaces and obtain a constructive descriptions of the Lipschitz classes of functions defined by the fractional order modulus of smoothness, in particular , in 2009 , A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces [6] , improved converse theorems of trigonometric approximation in variable exponent Lebesgue spaces with some Mucken-houpt weights , 1n 2009 , Operators of Harmonis Analysis in weighted spaces with non-standard growth [7] , e develop a certain variant of Rubio de Francia's extrapolation theorem. This extrapolation theorem is applied to obtain the boundedness in such spaces of various operators of harmonic analysis, such as maximal and singular operators, potential operators, Fourier multipliers, dominants of partial sums of trigonometric Fourier series and others, in weighted Lebesgue spaces with variable exponent. There are also given their vectorvalued analogues.,in 2010 , Sharp Jackson and converse theorems of trigonometric approximation in weighted Lebesgue spaces [8] , prove that improved Jackson type direct theorem of trigonometric polynomial approximation in Lebesgue spaces with Muckenhoupt weights with respect to fractional order moduli of smoothness holds. In addition, we obtain sharp converse and Marchaud inequalities of trigonometric approximation of functions and its fractional derivatives in these weighted Lebesgue spaces

2.Auxilary lemma In this section, we will present lemma that is needed in our main results

Lemma 2.1 If $\varphi^{-t_0} \in B_{\left(\frac{t(.)}{t_0}\right)^{\prime}}$ and $t \in T^{log}$ for some $t_0 \in (1, t_*)$

$then\ \varphi \in B_{t(\,.\,)}$.

Proof :-Using the Extrapolation theorem 3.2 of [7] we get that Hardy Little-wood maximum operator $\mathcal{U}$ is bounded in $\mathcal{L}_{\varphi}^{t(.)}$ this means [1] that $\varphi \in B_{t(.)}$.

3.Main results In this part, the theorems we need are presented in order to reach important results that we need later

Theorem 3.1 If $\varphi^{-t_0} \in B_{\left(\frac{t(.)}{t_0}\right)^{`}}$ and $t \in T^{log}$ for some

$T_0 \in (1, t_*), m \in \mathbb{N}, s \in \mathbb{R}^+, \imath := \min\{2, t_*\}$ *and*

$g \in \mathcal{L}_{\varphi}^{t(.)}$, *there isa*positive constant d_8 *it just depends t and s as in*

$$\jmath_s\left(g, \frac{1}{m}\right)_{p,\mu} \leq \frac{d_8}{m^s}\left\{\sum_{n=1}^{m} n^{\imath s-1} \mathcal{E}_{n-1}^{\imath}(g)_{p,\mu}\right\}^{\frac{1}{\imath}}$$

Then $y^{\imath}$ *is convex for* $\imath = \min\{2, t_*\}$ *we do have*

$$\left(n\, n^{s-1} \mathcal{E}_n(g)_{p,\mu}\right)^{\imath} - \left((n-1)\, n^{s-1} \mathcal{E}_n(g)_{p,\mu}\right)^{\imath}$$

$$\leq \left(\sum_{\omega=1}^{n} \omega^{s-1} \mathcal{E}_{\omega}(g)_{p,\mu}\right)^{\imath} - \left(\sum_{\omega=1}^{n-1} \omega^{s-1} \mathcal{E}_{\omega}(g)_{p,\mu}\right)^{\imath}.$$

Now we will add the last inequalities $n = 1, 2, 3, \ldots$ to get

$$\sum_{n=1}^{m}\left\{\left(n\, n^{s-1} \mathcal{E}_n(g)_{p,\mu}\right)^{\imath} - \left((n-1)\, n^{s-1} \mathcal{E}_n(g)_{p,\mu}\right)^{\imath}\right\}$$

$$\leq \sum_{n=1}^{m}\left\{\left(\sum_{\omega=1}^{n} \omega^{s-1} \mathcal{E}_{\omega}(g)_{p,\mu}\right)^{\imath} - \left(\sum_{\omega=1}^{n-1} \omega^{s-1} \mathcal{E}_{\omega}(g)_{p,\mu}\right)^{\imath}\right\}$$

Also

$$\left\{\sum_{n=1}^{m} n^{i\,s-1}\ \mathcal{E}_{n-1}^{i}\,(g)_{p,\mu}\right\}^{\frac{1}{i}} \le\ 2\sum_{n=1}^{m} n^{s-1}\ \mathcal{E}_{n-1}^{i}\,(g)_{p,\mu}\ .$$

The last inequality indicates that the theorem 3.1 it is an improvement of the converse theorem , the theory 3.1 of inequality has given highly accurate results

If

$$\mathcal{E}\,(g)_{p,\mu} = \mathcal{Y}\left(\frac{1}{m^s}\right)\ ,\ n\ \in\ \mathbb{N}$$

Then

$$\mathcal{j}_s\left(g\,,\frac{1}{m}\right)_{p\,,\mu} = \mathcal{Y}\left(\frac{1}{m^s}\left\|\log\frac{1}{m}\right\|_{p\,,\mu}\right)$$

By theorem 3.1 we get

$$\mathcal{j}_s\left(g\,,\frac{1}{m}\right)_{p\,,\mu} = \mathcal{Y}\left(\frac{1}{m^s}\left\|\log\frac{1}{m}\right\|_{p\,,\mu}\right)^{\frac{1}{i}}\ .$$

Proof:- In the beginning , using Lemma 2.1 by condition of theorem 3.1 the condition $\varphi\ \in B_{t\,(.\,)}$ satisfy . We notice that on the other hand , it is known as

$\mathcal{S}_{x\,,f_1\,,f_2\,,\ldots\ldots f_{[s]}}^{s}\ g\ :=\ \prod_{j=1}^{[s]}\left(J-\mathcal{S}_{,f_j}\right)(J-\mathcal{S}_x)^{s-[s]}\ g$ has Fourier series

$$\mathcal{S}_{x\,,f_1\,,f_2\,,\ldots\ldots f_{[s]}}^{s}\ g\,(.)\sim\sum_{n=-\infty}^{\infty}\left(1-\frac{\sin n\,x}{nx}\right)^{s-[s]}\left(1-\frac{\sin n\ f_1}{n\ f_1}\right)\ldots\left(1-\frac{\sin n\ f_{[s]}}{n\ f_{[s]}}\right)d_n\,e^{j\,n}$$

Also

$$\begin{aligned}\mathcal{S}_{x\,,f_1\,,f_2\,,\ldots\ldots f_{[s]}}^{s}\ g\,(.) &= \mathcal{S}_{x\,,f_1\,,f_2\,,\ldots\ldots f_{[s]}}^{s}\left(g\,(.)-W_{2^{u-1}}\,(.\,,g)\right)\\ &+\ \mathcal{S}_{x\,,f_1\,,f_2\,,\ldots\ldots f_{[s]}}^{s}\,W_{2^{u-1}}\,(.\,,g)\,.\end{aligned}$$

By $\mathcal{E}_m(g)_{p,\mu} \downarrow 0$ we have

$$\left\| \mathcal{S}^s_{x,f_1,f_2,\ldots,f_{[s]}} \left(g(.) - W_{2^{u-1}}(.\,,g) \right) \right\|_{p,\mu} \le d_{14}(s,t) \, \| g(.) - W_{2^{u-1}}(.\,,g) \|_{p,\mu}$$

$$\le d_{15}(s,t)\, \mathcal{E}_{2^{u-1}}(g)_{p,\mu}$$

$$\le \frac{d_{16}(s,t)}{m^s} \left\{ \sum_{n=1}^{m} n^{i\,s-1} \mathcal{E}^{i}_{n-1} (g)_{p,\mu} \right\}^{\frac{1}{i}} .$$

on the other hand through 3.2 we conclude

$$\left\| \mathcal{S}^s_{x,f_1,f_2,\ldots,f_{[s]}} W_{2^{u-1}}(.\,,g) \right\|_{p,\mu} \le d_{17}(s,t) \left\| \sum_{\omega=1}^{u} |Q_\omega|^2 \right\|_{p,\mu}$$

Where

$$Q_\omega := \sum_{|n|=2^{\omega-1}}^{2^{\omega-1}} \left(1 - \frac{\sin n\,x}{nx}\right)^{s-[s]} \left(1 - \frac{\sin n\, f_1}{n\, f_1}\right) \ldots \left(1 - \frac{\sin n\, f_{[s]}}{n\, f_{[s]}}\right) d_n\, e^{j\,n\,y}$$

By [6]

$$\left\| \sum_{\omega=1}^{u} |Q_\omega|^2 \right\|_{p,\mu} \le \left\{ \sum_{\omega=1}^{u} \|Q_\omega\|^{i_1}_{p,\mu} \right\}^{\frac{1}{i_1}} .$$

The estimate $\|Q_\omega\|_{p,\mu}$ then

$$\|Q_\omega\|_{p,\mu} = \left\| \sum_{|n|=2^{\omega-1}}^{2^{\omega-1}} \left[|n|^s \sum_{|n|=2^{\omega-1}}^{2^{\omega-1}} \left(1 - \frac{\sin n\,x}{nx}\right)^{s-[s]} \left(1 - \frac{\sin n\, f_1}{n\, f_1}\right) \ldots \left(1 - \frac{\sin n\, f_{[s]}}{n\, f_{[s]}}\right) \right] . \left[\frac{1}{|n|^s} d_n\, e^{j\,n\,y} \right] \right\|_{p,\mu}$$

Using Abel's transformation we conclude

$$\|Q_\omega\|_{p,\mu} \leq \sum_{|n|=2^{\omega-1}}^{2^{\omega-2}} \Bigg| n^s \left(1-\frac{\sin n\, x}{nx}\right)^{s-[s]} \left(1-\frac{\sin n\ f_1}{n\ f_1}\right) \dots \left(1-\frac{\sin n\ f_{[s]}}{n\ f_{[s]}}\right) - (n+1)^s \left(1-\frac{\sin(n+1)\,x}{(n+1)x}\right)^{s-[s]} \left(1-\frac{\sin(n+1)\ f_1}{(n+1)\ f_1}\right) \dots \left(1-\frac{\sin(n+1)\ f_{[s]}}{(n+1)\ f_{[s]}}\right)\Bigg| \cdot \left\| \sum_{|\ell|=2^{\omega-1}}^{n} \frac{1}{|\ell|^s} \left|d_\ell\, e^{j\,\ell\, y}\right| \right\|_{p,\mu}$$

$$+ \left|(2^\omega-1)^s \left(1-\frac{\sin(2^\omega-1)x}{(2^\omega-1)x}\right)^{s-[s]} \left(1-\frac{\sin(2^\omega-1)f_1}{(2^\omega-1)f_1}\right) \dots \left(1-\frac{\sin(2^\omega-1)\ f_{[s]}}{(2^\omega-1)\ f_{[s]}}\right)\right| \left\| \sum_{|\ell|=2^{\omega-1}}^{2^\omega-1} \frac{1}{|\ell|^s} \left|d_\ell\, e^{j\,\ell\, y}\right| \right\|_{p,\mu}$$

With us

$$\left\| \sum_{|\ell|=2^{\omega-1}}^{2^\omega-1} \frac{1}{|\ell|^s} \left|d_\ell\, e^{j\,\ell\, y}\right| \right\|_{p,\mu} \leq \frac{d_{18}(s,t)}{|2^{\omega-1}|^s} \left\| \sum_{|\ell|=2^{\omega-1}}^{2^\omega-1} \left|d_\ell\, e^{j\,\ell\, y}\right| \right\|_{p,\mu}$$

$$= \frac{d_{18}(s,t)}{|2^{\omega-1}|^s} \left\| \sum_{|\ell|=2^{\omega-1}}^{2^\omega-1} e^{-j\arg\left(d_\ell\, e^{j\,\ell\, y}\right)} \left|d_\ell\, e^{j\,\ell\, y}\right| \right\|_{p,\mu}$$

$$= \frac{d_{18}(s,t)}{|2^{\omega-1}|^s} \left\| \sum_{|\ell|=2^{\omega-1}}^{2^\omega-1} \left|d_\ell\, e^{j\,\ell\, y}\right| \right\|_{p,\mu} \leq \frac{d_{19}(s,t)}{2^{\omega s}}\, \mathcal{E}_{2^{\omega-1}-1}\,(g)_{p,\mu}$$

And also

$$\left\| \sum_{|\ell|=2^{\omega-1}}^{2^{\omega}-1} \frac{1}{|\ell|^{s}} \left| d_{\ell}\, e^{j\ell y} \right| \right\|_{p,\mu} \le \frac{d_{19}(s,t)}{2^{\omega s}} \mathcal{E}_{2^{\omega-1}-1}(g)_{p,\mu}$$

And because $y^{s}(1-\frac{\sin y}{y})^{s}$ not decreasing also $\left(1-\frac{\sin y}{y}\right) \le y$ for $y>0$

we get

$$\| Q_{\omega} \|_{p,\mu} \le \frac{d_{20}(s,t)\, 2^{-\omega s}}{x^{s-[s]} f_{1} \ldots f_{[s]}} \left[\sum_{|n|=2^{\omega-1}}^{2^{\omega}-2} \left| (nx)^{s-[s]} \left(1-\frac{\sin n x}{nx}\right) (n f_{1}) . \left(1 - \frac{\sin n f_{1}}{n f_{1}}\right) \ldots (n f_{[s]}) \left(1-\frac{\sin n f_{[s]}}{n f_{[s]}}\right) - ((n+1)x)^{s-[s]} \left(1-\frac{\sin(n+1)x}{(n+1)x}\right)^{s-[s]} . ((n+1) f_{1}) \left(1 - \frac{\sin (n+1) f_{1}}{(n+1) f_{1}}\right) \ldots (n+1) f_{[s]} \left(1-\frac{\sin(n+1) f_{[s]}}{(n+1) f_{[s]}}\right) \right| \right].$$

.

$\mathcal{E}_{2^{\omega-1}-1}(g)_{p,\mu}$ +

$d_{20}(s,t)\, 2^{-\omega s} \left| ((2^{\omega}-1)x)^{s-[s]} \left(1-\frac{\sin(2^{\omega}-1)x}{(2^{\omega}-1)x}\right)^{s-[s]} . (2^{\omega}-1) f_{1} \left(1 - \frac{\sin(2^{\omega}-1) f_{1}}{(2^{\omega}-1) f_{1}}\right) \ldots (2^{\omega}-1) f_{[s]} \left(1-\frac{\sin(2^{\omega}-1) f_{[s]}}{(2^{\omega}-1) f_{[s]}}\right) \right|.$

$$\mathcal{E}_{2^{\omega-1}-1}(g)_{p,\mu} \le d_{21}(s,t) \left(1-\frac{\sin(2^{\omega}-1)x}{(2^{\omega}-1)x}\right)^{s-[s]} \left(1-\frac{\sin(2^{\omega}-1) f_{1}}{(2^{\omega}-1) f_{1}}\right) \ldots \left(1 - \frac{\sin(2^{\omega}-1) f_{[s]}}{(2^{\omega}-1) f_{[s]}}\right) \mathcal{E}_{2^{\omega-1}-1}(g)_{p,\mu}$$
$$\le d_{22}(s,t) .\, 2^{\omega s} x^{s-[s]} f_{1} \ldots f_{[s]}\, \mathcal{E}_{2^{\omega-1}-1}(g)_{p,\mu}$$

So

$$\| Q_\omega \|_{p,\mu} \leq d_{22}(s,t) \,.\, 2^{\omega s} \, x^{s-[s]} \, f_1 \ldots \, f_{[s]} \, \mathcal{E}_{2^{\omega-1}-1}(g)_{p,\mu} \,.$$

And therefore

$$\left\| \mathcal{S}^s_{x,f_1,f_2,\ldots,f_{[s]}} \, W_{2^{u-1}}(\,.\,,g) \right\|_{p,\mu}$$

$$\leq d_{23}(s,t) \, x^{(s-[s])} \, f_1\,,f_2\,,\ldots,,f_{[s]} \left\{ \sum_{\omega=1}^{u} 2^{2\,\omega s i} \, \mathcal{E}^{i}_{2^{\omega-1}-1}(g)_{p,\mu} \right\}^{\frac{1}{i}}$$

$$\leq d_{24}(s,t) \, x^{(s-[s])} \, f_1\,,f_2\,,\ldots,,f_{[s]} \left\{ 2^{\,i_1 s} \, \mathcal{E}^{i}_{0}(g)_{p,\mu} \right\}^{\frac{1}{i}}$$

$$+ \, d_{25}(s,t) \, x^{(s-[s])} \, f_1\,,f_2\,,\ldots,,f_{[s]} \left\{ \sum_{\omega=1}^{u} \sum_{n=2^{\omega-2}}^{2^{\omega-1}-1} n^{i\,s-1} \, \mathcal{E}^{i}_{n-1}(g)_{p,\mu} \right\}^{\frac{1}{i}}$$

$$\leq d_{26}(s,t) \, x^{(s-[s])} \, f_1\,,f_2\,,\ldots,,f_{[s]} \left\{ \sum_{n=1}^{2^{u-1}-1} n^{i\,s-1} \, \mathcal{E}^{i}_{n-1}(g)_{p,\mu} \right\}^{\frac{1}{i}} .$$

The recent inequality means

$$\jmath_s\left(g\,,\frac{1}{m}\right)_{p,\mu} \leq \frac{d_{27}(s,t)}{m^s} \left\{ \sum_{n=1}^{m} n^{i\,s-1} \, \mathcal{E}^{i}_{n-1}(g)_{p,\mu} \right\}^{\frac{1}{i}}$$

Theorem 3.2 Under theoretical conditions **Theorem 3.1 , there are** d_{12} , $d_{13} > 0$

just depend on $s \; and \; t$ such that

$$
\mathrm{d}_{12} \left\| \left(\sum_{\omega=n}^{\infty} | \triangle_\omega |^2 \right)^{\frac{1}{2}} \right\|_{p,\mu} \le \left\| \sum_{|\omega|=2^{n-1}}^{\infty} d_n e^{j n y} \right\|_{p,\mu}
$$

$$
\le \mathrm{d}_{13} \left\| \left(\sum_{\omega=n}^{\infty} | \triangle_\omega |^2 \right)^{\frac{1}{2}} \right\|_{p,\mu} \qquad (2.1)
$$

where

$$
\triangle_\omega := \triangle_\omega (y, g) := \sum_{|n|=2^{\omega-1}}^{2^{\omega}-1} d_n e^{j n y} .
$$

The proof of this theorem same theorem proof in [7] Littlewood – Paley

Theorem 3.3 Under theoretical conditions **Theorem 3.1**

$$
\sum_{c=1}^{\infty} c^{i \beta-1} \mathcal{E}_c^i (g)_{p,\mu} < \infty \qquad (1,8)
$$

$\mathcal{E}_m (g^{(\beta)})_{p,\mu} \le d_{10} \left(m^\beta \mathcal{E}_m (g)_{p,\mu} + \left\{ \sum_{n=m+1}^{\infty} n^{\beta i-1} \mathcal{E}_c^i (g)_{p,\mu} \right\}^{\frac{1}{i}} \right)$ satisfy corollary

Proof :- Suppose $\mathcal{F}_m$ be a polynomial of class $\mathcal{M}_m$ such that

$\mathcal{E}_m (g)_{p,\mu} = \|g - \mathcal{F}_m\|_{p,\mu}$ *then*

$\mathcal{A}_0 (y) := \mathcal{F}_1 (y) - \mathcal{F}_0 (y)$; $\mathcal{A}_n (y) := \mathcal{F}_{2^n} (y) - \mathcal{F}_{2^{n-1}} (y)$,

$n = 1, 2, 3, \ldots$

Now

$$
\mathcal{F}_{2^M} (y) = \mathcal{F}_0 (y) + \sum_{n=0}^{M} \mathcal{A}_n (y), \quad M = 0, 1, 2, \ldots .
$$

$\forall \varepsilon > 0, \exists \mathcal{D} \in \mathbb{N}$ *by* (2.1) *such that*

$$\sum_{n=2^{\mathcal{D}}}^{\infty} n^{i\,\beta-1}\ \mathcal{E}_n^{i}\,(g)_{p,\mu} < \varepsilon \qquad (2.2)$$

By fractional Bernstein's

$$\left\|\mathcal{F}_{m}^{(\beta)}\right\|_{p,\mu} \le d_{28}\,(\beta,t)\,m^{\beta}\,\|\mathcal{F}_m\|_{p,\mu}\ , \qquad \beta \in \mathbb{R}^{+}$$

But we have

$$\left\|\mathcal{A}_n^{(\beta)}\right\|_{p,\mu} \le d_{29}\,(\beta,t)\,2^{n\beta}\,\|\mathcal{A}_n\|_{p,\mu} \le d_{30}\,(\beta,t)\,2^{n\beta}\,\mathcal{E}_{2^{n-1}}\,(g)_{p,\mu}\ , n \in \mathbb{N}.$$

It is easy for us to see this from the other side

$$2^{n\beta}\,\mathcal{E}_{2^{n-1}}\,(g)_{p,\mu} \le d_{31}\,(\beta,t)\left\{\sum_{\omega=2^{n-2}+1}^{2^{n-1}} \omega^{i\,\beta-1}\,\mathcal{E}_{\omega}^{i}\,(g)_{p,\mu}\right\},\quad n = 2,3,4,\ldots.$$

With respect to positive integers $C < M$

$$\mathcal{F}_{2^M}^{(\beta)}\,(y) - \mathcal{F}_{2^C}^{(\beta)}\,(y) = \sum_{n=C+1}^{M} H_n^{(\beta)}\,(y)\ , \qquad y \in \mathcal{F}$$

And later , if it is large enough, we get from (2 . 2)

$$\left\|\mathcal{F}_{2^M}^{(\beta)}\,(y) - \mathcal{F}_{2^C}^{(\beta)}\,(y)\right\|_{p,\mu} \le \sum_{n=C+1}^{M}\left\|\mathcal{A}_n^{(\beta)}\,(y)\right\|_{p,\mu}$$

$$\le d_{31}\,(\beta,t)\sum_{n=C+1}^{M} 2^{n\beta}\,\mathcal{E}_{2^{n-1}}\,(g)_{p,\mu}$$

$$\le d_{32}\,(\beta,t)\sum_{n=C+1}^{M}\left\{\sum_{\omega=2^{n-2}}^{2^{n-1}} \omega^{i\,\beta-1}\,\mathcal{E}_{\omega}^{i}\,(g)_{p,\mu}\right\}^{\frac{1}{i}}$$

$$\leq d_{33}(\beta,t)\left\{\sum_{\omega=2^{C-2}+1}^{2^{M-1}} \omega^{i\beta-1}\, \mathcal{E}_{\omega}^{i}(g)_{p,\mu}\right\}^{\frac{1}{i}} \leq d_{33}(\beta,t)\,\varepsilon^{\frac{1}{i}}.$$

$\left\{\mathcal{F}_{2^M}^{(\beta)}\right\}$ *is Cauchy sequence in* $\mathcal{L}_{\varphi}^{t(.)}$ *then there is b* $\sigma \in \mathcal{L}_{\varphi}^{t(.)}$ *hold*

$$\left\|\mathcal{F}_{2^M}^{(\beta)} - \sigma\right\|_{p,\mu} \to 0, \qquad as\ M \to \infty$$

from the other side we get

$$\left\|\mathcal{F}_{2^M}^{(\beta)} - g^{(\beta)}\right\|_{p,\mu} \to 0, \qquad as\ M \to \infty.$$

$$g^{(\beta)} = \sigma\ b.a. \quad then\ g \in K_{t(.),\varphi}^{\beta}.$$

Now

$$\mathcal{E}_m(g^{(\beta)})_{p,\mu} \leq \left\|g^{(\beta)} - W_m\, g^{(\beta)}\right\|_{p,\mu}$$

$$\leq \left\|W_{2^{u+2}}\, g^{(\beta)} - W_m\, g^{(\beta)}\right\|_{p,\mu} + \left\|\sum_{c=u+2}^{\infty}\left[W_{2^{c+2}}\, g^{(\beta)} - W_{2^c}\, g^{(\beta)}\right]\right\|_{p,\mu} \qquad (2.3)$$

We use the fractional Bernstein's inequality we get for $2^u < m < 2^{u+1}$

$$\left\|W_{2^{u+2}}\, g^{(\beta)} - W_m\, g^{(\beta)}\right\|_{p,\mu} \leq d_{35}(\beta,t)2^{(u+2)}\,\mathcal{E}_m(g)_{p,\mu} \quad (2.4)$$

$$\leq d_{36}(\beta,t)\, m^{\beta}\, \mathcal{E}_m(g)_{p,\mu}.$$

By (2 . 1) we get

$$\left\|\sum_{c=u+2}^{\infty}\left[W_{2^{c+2}}\, g^{(\beta)} - W_{2^c}\, g^{(\beta)}\right]\right\|_{p,\mu}$$

$$\leq d_{37}(\beta,t)\left\|\sum_{c=u+2}^{\infty}\left|\sum_{|n|=2^{C}+1}^{2^{C+1}}((j\,n)_{p,\mu})^{\beta}\ d_n\ e^{j\,n\,y}\right|^2\right\|_{p,\mu}$$

And

$$\left\| \sum_{c=u+2}^{\infty} [W_{2^{c+2}}\, g^{(\beta)} - W_{2^c}\, g^{(\beta)}] \right\|_{p,\mu}$$

$$\leq d_{38}(\beta, t) \left(\sum_{c=u+2}^{\infty} \left\| \sum_{|n|=2^c+1}^{2^{c+1}} ((j\,n)_{p,\mu})^{\beta}\, d_n\, e^{j\,n\,y} \right\|_{p,\mu}^{\frac{1}{i}} \right).$$

We put

$$\|Q_n^*\|_{p,\mu} := \sum_{|n|=2^c+1}^{2^{c+1}} ((j\,n)_{p,\mu})^{\beta}\, d_n\, e^{j\,n\,y} = \sum_{n=2^c+1}^{2^{c+1}} n^{\beta} 2\, Re\,(d_n\, e^{j\,(n\,y+\beta\pi/2)})$$

And we have

$$\|Q_n^*\|_{p,\mu} := \left\| \sum_{n=2^c+1}^{2^{c+1}} n^{\beta}\, H_n(y) \right\|_{p,\mu}$$

Such that $H_n\,(y = \; 2\,Re\,(d_n\; e^{\,j\,(n\,y+\,\beta\pi/2\,)})$.Using Abel’s transformation we get

$$\|Q_n^*\|_{p,\mu} \leq \sum_{n=2^c+1}^{2^{c+1}-1} \left| n^{\beta} - (n+1)^{\beta} \right| \left\| \sum_{\ell=2^c+1}^{n} H_{\ell}(y) \right\|_{p,\mu}$$

$$+ \left| (2^{c+1})^{\beta} \right| \left\| \sum_{\ell=2^c+1}^{2^{c+1}-1} \mathrm{H}_{\ell}(\mathrm{y}) \right\|_{p,\mu}$$

$2^c + 1 \leq n \leq \; 2^{c+1} \quad (c \in \mathbb{N})$ *we have*

$$\left\| \sum_{\ell = 2^c+1}^{n} H_\ell(y) \right\|_{p,\mu} \le d_{39}(\beta, t)\, \mathcal{E}_{2^c}(g)_{p,\mu}$$

Since

$$(n+1)^\beta - n^\beta \le \begin{cases} \beta\,(n+1)^{\beta-1} & ,\beta \ge 1, \\ \beta\, n^{\beta-1} & ,0 \le \beta < 1, \end{cases}$$

Then

$$\|Q_n^*\|_{p,\mu} \le d_{40}(\beta, t)\, 2^{c\beta}\, \mathcal{E}_{2^c-1}(g)_{p,\mu}.$$

Also

$$\left\| \sum_{c=u+2}^{\infty} \left[W_{2^{c+1}}\, g^{(\beta)} - W_{2^c}\, g^{(\beta)} \right] \right\|_{p,\mu}$$

$$\le d_{41}(\beta, t) \left\{ \sum_{c=u+2}^{\infty} 2^{c\beta i}\, \mathcal{E}_{2^n-1}^{i}(g)_{p,\mu} \right\}^{\frac{1}{i}}$$

$$\le d_{42}(\beta, t) \left\{ \sum_{c=u+2}^{\infty} n^{i\beta-1}\, \mathcal{E}_n^{i}(g)_{p,\mu} \right\}^{\frac{1}{i}}$$

Corollary 3.4 If

$s, \ell \in \mathbb{R}^+,\ s < \ell$ *and*

$0 < x \le \frac{1}{2}$, then *there is* a positive constant d_9 depending only on s, ℓ *and* t

Such that

$$\jmath_s(g, x)_{t(.),\varphi} \le d_9\, x^s \left\{ \int_x^1 \left[\frac{\jmath_\ell(g, h)_{t(.),\varphi}}{h^s} \right]^i \frac{dh}{h} \right\}^{\frac{1}{i}}$$

Corollary 3.5 Under theoretical conditions **Theorem 3.1,** then *there is* a positive constant d_{11} depending only on s , β *and* t

Such that

$$\jmath_s\left(g^{(\beta)},\frac{1}{m}\right)_{p,\mu} \leq d_{11}\left(\frac{1}{m^s}\left(\sum_{n=1}^{m} n^{i(s+\beta)-1}\ \mathcal{E}_c^i(g)_{p,\mu}\right)^{\frac{1}{i}} + \left(\sum_{n=m+1}^{m} n^{\beta i-1}\ \mathcal{E}_c^i(g)_{p,\mu}\right)^{\frac{1}{i}}\right)$$

For $m \in \mathbb{N}$ *and* $\beta, s \in \mathbb{R}^+$

References

[1] P. H¨ast¨o and L. Diening, Muckenhoupt weights in variable exponent spaces, Preprint, Albert Ludwings Universit¨at Freiburg, Mathematische Fakult¨at,(2014) http://www.helsinki.fi/˜ pharjule/varsob/publications.shtml.

[2] V. Kokilashvili and S.G. Samko, Singular integrals weighted Lebesgue spaces with variable exponent, Georgian M. J. 10 (2003), No:1, 145-156.

[3] V. Kokilashvili and Y.E. Yildirir, On the approximation in weighted Lebesgue spaces, Proc. A. Razmadze Math. Inst. 143 (2007), 103-113.

[4] V. Kokilashvili and Y.E. Yildirir, On the approximation in weighted Lebesgue spaces, Proc. A. Razmadze Math. Inst. 143 (2007), 103-113.

[5] R. Akgun and D.M. Israfilov, Approximation and moduli of fractional orders in Smirnov- Orlicz classes, Glas. Mat. Ser. III 43(63) (2008), no. 1, 121-136.

[6] --, A refined inverse inequality of approximation in weighted variable exponent Lebesgue spaces, Proc. A. Razmadze Math. Inst. 151 (2009), 134-138