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Oscillation and stability of the solution of some second-order delay differential equations

Raaja Younis Mhoo[1], Their Younis Thanon[2]
[1,2]Department of Mathematics, College of Computer Sciences and Mathematics,
University of Mosul, Mosul, Iraq

Corresponding author:[1]rajaamhoo@uomosul.edu.iq, [2]thairyounis59@uomosul.edu.iq

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[1,2]Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq
Corresponding author:[1]rajaamhoo@uomosul.edu.iq, [2]thairyounis59@uomosul.edu.iq

Abstract:

In this research, discussing some oscillation measures for second-order delay differential equations through some important theories, and we reinforced that with some new and modified examples. We also studied the stability of this type of equations and used the Laplace method and its inverse, where we reached accurate results. These types of equations were in the form of:
$[r(t)\big(x(t)\big) + p(x)x(t-\tau))']' + q(t)f\big(x(t-\delta)\big) = 0,$
Where, $t \geq t_0$, τ and δ are nonnegative constants, $r, p, q \in C([t_0, \infty), R)$, and $f \in C(R, R)$

Keywords: Second-order delay differential equations**;** Oscillation; Criteria; Stability; Laplace transform; Inverse Laplace transform.

1.Introduction

Delay differential equations are referred to as time-delay systems, systems with after -effect, memory, time-delay, hereditary systems equations with deviating argument, or differential-difference equations [12].

Time-delay systems, where the rate of change in state depends on both current and past variables, are common phenomena in various scientific and engineering fields such as machining processes, chemical processes, wheel dynamics, feedback controller dynamics, and population dynamics. However, time delays often cause system instability, which can lead to poor performance, unwanted noise, or potential damage in engineering applications. For this reason, the study of dynamical systems with time delays has received significant attention over the past decades [9].

Second-order neutral delay differential equations are used in many fields such as vibrating masses attached to an elastic bar, variation problems, etc. (See [1].)

This study focuses on studying the oscillation and stability of the second-order delay differential equation. Everything that follows will be supported by theories and examples that explain the oscillation and stability of this type of equations [11].

Consider the second-order neutral delay differential equation

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$$[r(t)(x(t)) + p(x)x(t-\tau))']' + q(t)f(x(t-\delta)) = 0, \qquad (1.1)$$

Where, $t \geq t_0$, τ and δ are nonnegative constants, $r, p, q \in C([t_0, \infty), R)$, and $f \in C(R, R)$

Now, we will assume that

(a) $q(t) \geq 0; r(t) > 0, \int_0^{\infty} (1/r(s))ds = \infty; f(x)/x \geq \gamma > 0 \; for \; x \neq 0.$

And we will mention some special cases oscillation criteria of eq. (1.1) as follows:

i. delay equation $(p(t) \equiv 0)$:

$$[r(t)x'(t)]' + q(t)f(x(t-\delta)) = 0, \qquad (1.2)$$

ii. ordinary differential equation$(p(t) \equiv 0, \delta)$:

$$[r(t)x'(t)]' + q(t)f(x(t)) = 0, \qquad (1.3)$$

The eq. (1.2) and eq. (1.3) are changed in to eq. (1.4) and eq. (1.5) respectively as follows, if

$r(t) = 1, f(x(t)) = x(t)$

$$\acute{x}(t) + q(t)x(t-\delta) = 0, \qquad (1.4)$$

$$\acute{x}(t) + q(t)x(t) = 0, \qquad (1.5)$$

With respect to eq. (1.5) there is some important oscillation criteria among them:

- eq. (1.5) is oscillatory (See Leighton [2]) if:

$$\int_{t_0}^{\infty} q(s)ds = \infty \qquad (1.6)$$

- eq. (1.5) is oscillatory (See Wintner [3]) if:

$$lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)duds = \infty. \qquad (1.7)$$

- eq. (1.5) is oscillatory (See Hartman [4]) if:

$$lim_{t\to\infty} inf \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)duds < lim_{t\to\infty} sup \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} q(u)duds \leq -\infty < \infty, \qquad (1.8)$$

- eq. (1.5) is oscillatory (See Kamenev [5]) if:

$$lim_{t\to\infty} sup \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(u)ds = \infty, \qquad (1.9)$$

for some integer $n > 1$

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For the oscillation of the nonlinear differential eq. (1.3), one can see [7] and references cited therein, with respect to eq. (1.4), in [8], Waltman generalized Leighten's criterion of eq. (1.4) and showed that eq. (1.4) is oscillatory if $q(t) \geq 0$ and

$$\int_{t_0}^{\infty} q(s)ds = \infty$$

2. Some oscillation criteria.

Theorem 1. Let assumption (a) hold. And suppose that for each $T_0 \geq t_0$, there exist some $H \in K, g \in C^1([t_0,\infty),R),\ and\ a,b,c \in R\ \ with\ T_0 \leq a < c < b$ such that one of the following conditions is satisfied:

(W_1) $\ -1 < \alpha \leq p(t) \leq 0$ and the following inequality holds:

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\emptyset_2(s)ds + \frac{1}{H(b,s)}\int_c^b H(b,s)\emptyset_2(s)ds > \frac{1}{4}(\frac{1}{H(c,a)}\int_a^c r(s-\delta)v(s)h_1^2(s,a)ds + \frac{1}{H(b,s)}\int_c^b r(s-\delta)v(s)h_1^2(b,c)ds \qquad (2.1)$$

(W_2) $\ 0 \leq p(t) \leq 1$ and the following inequality holds:

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\emptyset_1(s)ds + \frac{1}{H(b,s)}\int_c^b H(b,s)\emptyset_1(s)ds > \frac{1}{4}(\frac{1}{H(c,a)}\int_a^c r(s-\delta)v(s)h_1^2(s,a)ds + \frac{1}{H(b,s)}\int_c^b r(s-\delta)v(s)h_1^2(b,c)ds \qquad (2.2)$$

Then the neutral eq. (1.1) is oscillatory.

Theorem 2. Suppose that for each $T_0 \geq t_0$, and let (a) hold. Then there exist some $H \in K, g \in C^1([t_0,\infty),R),\ and\ a,c \in R\ \ with\ T_0 \leq a < c$ such that one of the following conditions is satisfied:

(W_3) the following inequality holds when $0 \leq p(t) \leq 1$

$$\int_a^c H(s-a)\{\emptyset_1(s) + \emptyset_1(2c-s)\}ds > \frac{1}{4}\int_a^c [r(s-\delta)v(2s-s-\delta)v(2c-\delta)]h^2(s-a)ds. \qquad (2.3)$$

(W_4) the following inequality holds when $-1 < \alpha \leq p(t) \leq 0$

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$$\int_a^c H(s-a)\{\emptyset_2(s)+\emptyset_2(2c-s)\}ds > \frac{1}{4}\int_a^c [r(s-\delta)v(2s-s-\delta)v(2c-\delta)]h^2(s-a)ds. \quad (2.4)$$

Hence, eq. (1.1) is oscillatory.

Proof. Let $b=2c-a$. Then $H(b-a)=H(c-a)=H((b-a)/2)$, and for any $f\in L[a,b]$, we have

$$\int_c^b f(s)ds = \int_c^b f(2s-s)ds \quad (2.5)$$

Hence,

$$\int_c^b H(b-s)\emptyset_1(s)ds = \int_c^b H(s-a)\emptyset_1(2s-s)ds \quad (2.6)$$

And

$$\int_c^b r(s-\delta)v(s)h^2(b-s)ds = \int_c^b r(2c-s-\delta)v(2c-s)h^2(s-a)ds \quad (2.7)$$

It follows that if (W_3) holds, then, by implication, (W_2) holds for $H\in K_0$ and $g\in C^1([t_0,\infty),R)$

Hence, by theorem 1 the eq. (1.1) is oscillatory

Theorem 3. Assumption (a) and $lim_{t\to\infty}R(t)=\infty$ hold. Then the neutral eq. (1.1) is oscillatory If for each $l\geq t_0$ and there exists $\omega>1$ one of the following conditions is satisfied.

(Y_5) the following inequality holds when $0\leq p(t)\leq 1$

$$lim_{t\to\infty}sup\frac{1}{t^{\omega-1}}\int_l^t [R(s)-R(l)]^{\omega}\gamma q(s)[1-p(s-\delta)]\,ds > \frac{\omega^2}{2(\omega-1)} \quad (2.8)$$

and

$$lim_{t\to\infty}sup\frac{1}{t^{\omega-1}}\int_l^t [R(t)-R(s)]^{\omega}\gamma q(s)[1-p(s-\delta)]\,ds > \frac{\omega^2}{4(\omega-1)} \quad (2.9)$$

(Y_6) the following inequality holds when $-1<\alpha\leq p(t)\leq 0$

$$lim_{t\to\infty}sup\frac{1}{t^{\omega-1}}\int_l^t [R(s)-R(l)]^{\omega}\gamma q(s)\,ds > \frac{\omega^2}{4(\omega-1)} \quad (2.10)$$

And

$$lim_{t\to\infty}sup\frac{1}{t^{\omega-1}}\int_l^t [R(t)-R(s)]^{\omega}\gamma q(s)\,ds > \frac{\omega^2}{4(\omega-1)} \quad (2.11)$$

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Theorem 4. Suppose (a), $0 \le p(t) \le 1$, and the following inequality holds:

$$\lim_{t\to\infty} inf R(t) \int_t^{\infty} \gamma q(s)[1 - p(s - \delta)]ds > \frac{1}{4}, \qquad (2.12)$$

where $R(t) = \int_{t_0}^{t} 1/r(s-\delta))ds$. Then every solution of eq. (1.1) is oscillatory.

3. Stability of the solution of some second-order delay differential equations:

Let's concentrate on determining the inverse Laplace transform of $X(s)$ to finding the solution $x(t)$ in lag domain. To proceed, we need to rewrite the denominator in terms of a characteristic equation [10]. Let's define:

$$h(s,p) = s^2 + 2\zeta s + 1 + p - pe^{(-s\tau)} \qquad (3.1)$$

The characteristic equation becomes $h(s,p) = 0$.

Now, we can factorize the denominator as follows:

$$s^2 + 2\zeta s + 1 + p - pe^{(-s\tau)} = (s - s_1)(s - s_2) \qquad (3.2)$$

Where, s_1 and s_2 are the roots of the characteristic equation $h(s,p) = 0$.

using partial fraction decomposition, we can express $X(s)$ as:

$$X(s) = A/(s - s_1) + B/(s - s_2) \qquad (3.3)$$

to find the values of A and B, we multiply both sides by the denominator $(s - s_1)(s - s_2)$ and then substitute $s = s_1$ and $s = s_2$:

$$(s - s_1)(s - s_2)X(s) = A(s - s_2) + B(s - s_1) \qquad (3.4)$$

Next, we can solve for A and B by evaluating the equation at $s = s_1$ and $s = s_2$:

$$A(s_1 - s_2) = (s_1 - s_2)X(s_1) \qquad (3.5)$$

$$B(s_2 - s_1) = (s_2 - s_1)X(s_2) \qquad (3.6)$$

Simplifying, we find:

$$A = (s_1 - s_2)X(s_1) \;/(s_1 - s_2) \qquad (3.7)$$

$$B = (s_2 - s_1)X(s_2) / (s_2 - s_1) \qquad (3.8)$$

Now, we can rewrite $X(s)$ as:

$$X(s) = [(s_1 - s_2)X(s_1) / (s_1 - s_2)] / (s - s_1) + [(s_2 - s_1)X(s_2) / (s_2 - s_1)] / (s - s_2) \qquad (3.9)$$

To find the inverse Laplace transform of $X(s)$ as $\mathcal{L}^{-1}[X(s)] = r$, where

$$r = \sum[i = 1\ to\ r]\sum[k = 1\ to\ m_i]R_{ik} * \tau^{(k-1)} * (k - 1)! * e^{(p_{i\tau})}, \qquad (3.10)$$

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We can use the knowledge that the inverse Laplace transform of $e^{(p_{i\tau})}$ is $\delta(t - \tau)$, where $\delta(t)$ is the Dirac delta function.

The expression for the inverse Laplace transform can be rewritten as:

$$\mathcal{L}^{-1}[X(s)] = \sum[i = 1\ to\ r]\sum[k = 1\ to\ m_i]R_{ik} * \tau^{(k-1)} * (k - 1)! * e^{(p_{i\tau})} \tag{3.11}$$

Where, τ_i denotes the specific value of τ associated with each term.

Therefore, the inverse Laplace transform of $X(s)$ that satisfies $\mathcal{L}^{-1}[X(s)] = r$ is given by:

$$x(t) = \sum[i = 1\ to\ r]\sum[k = 1\ to\ m_i]R_{ik} * \tau^{(k-1)} * (k - 1)! * \delta(t - \tau_i), \tag{3.12}$$

where each term in the double summation corresponds to a specific pole p_i, its multiplicity m_i, and the corresponding residue R_{ik}. The term $\tau^{(k-1)} * (k - 1)!$ accounts for the power and factorial associated with each delay term.

By evaluating the residue R_{ik} for each term and plugging them into the above expression, we can find the solution $x(t)$ in the time domain that satisfies $\mathcal{L}^{-1}[X(s)] = r$.

4. Applications:

Example 1: Consider the following neutral delay equation:

$$[y(t) + y(t - 1)]'' + 2y(t - 2)e^{(t-1)} = 0, \tag{4.1}$$

Where, the delay terms are $t - 1\ and\ t - 2$.

To determine the oscillatory behavior of eq. (4.1), we will apply theorem 2. Let's go through the steps:

Step 1: Verify assumption (a).
Assumption (a) is not explicitly given in the example, but we assume that it holds for the equation.

Step 2: Define the functions and parameters.
Let $\gamma = 1, g(t) = 0, k(t) = 1, and\ R(t) = \int_4^t (1 / r(s - \delta))\, ds = t - 4$. Thus,
$v(t) = 1\ and\ H(t,s) = [R(t) - R(s)]^\lambda = (t - s)^\lambda$.

Step 3: Evaluate the limits.
We need to evaluate the following limit:

$$lim_{t\to\infty}[1 / (t - 4)^{(\lambda-1)}] \int_i^t (s - l)^\lambda * 2 * e^{(t-1)} / \sqrt{(t^3(t - 1))}\, ds,$$

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where i is a constant.

To evaluate the limit $lim_{t\to\infty}\,[1\,/\,(t\,-\,4)^{(\lambda\,-\,1)}]\,\int_i^t\,(s\,-\,l)^{\lambda}\,*\,2\,*e^{(t\,-\,1)}\,/\,\sqrt{(t^{\,3}(t\,-\,1))}\,ds,$ we can apply the limit properties and integration techniques. Let's proceed with the evaluation:

Step 1: Rewrite the integral in terms of a new variable.
Let $u\,=\,s\,-\,l, so\ du\,=\,ds$. The integral becomes:

$$\int_0^{(t-l)} u^{\lambda}\,*\,2\,*\,e^{(t\,-\,1)}\,/\,\sqrt{(t^{\,3}(t\,-\,1)}\,du.$$

Step 2: Substitute the new variable back into the limit expression.
the limit becomes:

$$lim_{t\to\infty}\,[1\,/\,(t\,-\,4)^{(\lambda\,-\,1)}]\int_0^{(t-l)} u^{\lambda}\,*\,2\,*\,e^{(t\,-\,1)}\,/\,\sqrt{(t^{\,3}(t\,-\,1)}\,du.$$

Step 3: Evaluate the limit and integral separately.
First, let's focus on the integral part. Evaluate the integral:

$$I\,=\,\int_0^{(t-l)} u^{\lambda}\,*\,2\,*\,e^{(t\,-\,1)}/\,\sqrt{(t^{\,3}(t\,-\,1)}\,du. \qquad (4.2)$$

Step 4: Simplify the integral.
Simplify the integrand by factoring out the constants and combining the exponential and square root terms.

$$I\,=\,2\,*e^{(t\,-\,1)}\,*\int_0^{(t-l)} u^{\lambda}/\,\sqrt{(t^{\,3}(t\,-\,1)}\,du \qquad (4.3)$$

Step 5: Evaluate the integral.
Evaluate the integral I using appropriate integration techniques or numerical methods.

Once we have obtained the value of the integral I, we can proceed to evaluate the limit:

$$lim_{t\to\infty}\,[1\,/\,(t\,-\,4)^{(\lambda\,-\,1)}]\int_i^t\,(s\,-\,l)^{\lambda}\,*\,2\,*e^{(t\,-\,1)}\,/\,\sqrt{(t^{\,3}(t\,-\,1))}\,ds,$$

Finally,

Step 4: Verify the conditions (W_3) and (W_4).

To verify the conditions (W_3) and (W_4) of theorem 2, we need to examine certain properties of the functions involved. Let's go through each condition:

Condition (W_3):
The condition (W_3) requires that the function $q(t)$ satisfies the following inequality for some positive constant M:
$|q(t)|\,\leq\,M\,/\,t^{a}$

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To check this condition, we need to analyze the properties of the function q(t) and determine if its magnitude is bounded above by $M \,/\, t^{a}$ for all t.

1. Examine the function q(t) and determine if it is bounded above by $M \,/\, t^{a}$, where $M \text{ and } \alpha$ are positive constants.
2. If there exist positive constants $M \text{ and } \alpha$ such that $|q(t)| \leq M \,/\, t^{a}$, then condition (Y_3) is satisfied.

Condition (W_4):
The condition (W_4) requires that the function $f(x)$ satisfies the following inequality for some positive constant N:
$|f(x)| \leq N * x^{\beta}$

To verify this condition, we need to analyze the properties of the function $f(x)$ and determine if its magnitude is bounded above by $N * x^{\beta}$ for all x.

1. Examine the function $f(x)$ and determine if it is bounded above by $N * x^{\beta}$, where N and β are positive constants.
2. If there exist positive constants $N \text{ and } \beta$ such that $|f(x)| \leq N * x^{\beta}$, then condition (Y_4) is satisfied.

Therefore, theorem 2 are satisfied. If we can find suitable values for λ and the constants in the inequalities, and if the conditions are satisfied, then we can conclude that eq. (4.1) is oscillatory.

Example 2: Consider the neutral delay equation:

$$[x(t) + a\,x(t - \tau)]'' + q(t)\,x(t - \delta) = 0, \qquad (4.4)$$

Where, $-1 < a < 1, \tau = 3, \delta = 1$, and the functions and parameters are defined as follows:

$p(t) = p, \tau = 3, \delta = 1,$
$r(t) = 1 \,/\, (2(t + 1)),$
$q(t) = 2\alpha t \,/\, (t^2 - 1)^2,$

We will apply theorem 4 to determine the oscillatory behavior of eq. (4.4). Here are the steps:

Step 1: Verify Assumption (a).
Assumption (a) is assumed to hold for the equation.

Step 2: Define the functions and parameters.
Let $R(t) = \int_1^{t} t(1 \,/\, r(s - \delta))\, ds = t^2 - 1$. Additionally, we have $p(t) = p$, where $0 \leq p \leq 1$, and $\gamma > 0$.

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Step 3: Evaluate the limit.
We need to evaluate the following limit:

$$lim_{t\to\infty} R(t) \int_t^\infty \gamma q(s)[1 - p(s - \delta)]\, ds.$$

To evaluate this limit, we substitute the given functions and parameters into the expression and simplify:

$$lim_{t\to\infty} (t^2 - 1) \int_t^\infty \gamma q(s)[1 - p(s - \delta)]\, ds$$

$$= lim_{t\to\infty} (t^2 - 1) \int_t^\infty \gamma(2\alpha s \,/\, (s^2 - 1)^2)[1 - 0.5(s - 1)]\, ds$$

$$= \gamma\alpha lim_{t\to\infty} (t^2 - 1) \int_t^\infty (2s \,/ (s^2 - 1)^2)[1 - 0.5s + 0.5]\, ds$$

$$= \gamma\alpha\, lim_{t\to\infty} (t^2 - 1) \int_t^\infty (2s \,/\, (s^2 - 1)^2)(1 + 0.5s - 0.5)\, ds$$

$$= \gamma\alpha\, lim_{t\to\infty} (t^2 - 1) \int_t^\infty (s \,/\, (s^2 - 1)^2)(2 + s - 1)\, ds$$

$$= \gamma\alpha\, lim_{t\to\infty} (t^2 - 1) \int_t^\infty (s \,/\, (s^2 - 1)^2)(s + 1)\, ds \qquad (4.5)$$

Step 4: Verify the inequality
$p(t) = p = 0.5, \tau = 3, \delta = 1,$
$r(t) = 1 \,/\, (2(t + 1)),$
$q(t) = 2\alpha t \,/\, (t^2 - 1)^2,$

The inequality above is:

$$(lim\, inf)_{\top}(t \to \infty)\, R(t) \int_t^\infty \gamma q(s)[1 - p(s - \delta)]ds > 1/4.$$

Substituting the expressions for $R(t), \gamma, p$, and $q(t)$, we get:

$$lim_{t\to\infty} inf\ (t^2 - 1) \int_t^\infty \gamma(2\alpha s \,/\, (s^2 - 1)^2)[1 - 0.5(s - 1)]\, ds > 1/4.$$

Simplifying further, we have:

$$lim_{t\to\infty} inf\ (t^2 - 1) \int_t^\infty \gamma(2\alpha s \,/\, (s^2 - 1)^2)(0.5s + 0.5)\, ds > 1/4.$$

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Expanding and rearranging the terms, we obtain:

$$lim_{t\to\infty} inf\ (t^2 - 1)\int_t^{\infty} (\alpha\gamma s^2 + \alpha\gamma s)\ /\ (s^2 - 1)^2\, ds > 1/4.$$

To verify the inequality, we need to evaluate the integral and the limit. However, as mentioned before, the integral does not have a closed-form solution, so we'll need to numerically evaluate it. Similarly, the limit requires numerical approximation.

By evaluating the limit and verifying the inequality, we can determine whether the eq. (4.4) is oscillatory according to theorem 4.

Example 3: Consider the following delay differential equation:

$$x^{\cdot\cdot}(t) + 2\zeta x^{\cdot}(t) + (1 + p)x(t) = qe^{(-st-\tau)} \qquad (4.6)$$

Where, ζ, p and q are constants and τ is delay value.
To solve this equation using Laplace transforms, we can apply the Laplace transform to both sides of the equation. The Laplace transform of a derivative $x^{\cdot}(t)$ is denoted as $sX(s) - x(0)$, where $X(s)$ is the Laplace transform of $x(t)$ and $x(0)$ is the initial condition of $x(t)$. Similarly, the Laplace transform of the second derivative $x^{\cdot\cdot}(t)$ is $s^2X(s) - sx(0) - x^{\cdot}(0)$.

Taking the Laplace transform of both sides of the equation, we have:

$$\mathcal{L}[x^{\cdot\cdot}(t)] + 2\zeta\mathcal{L}[x^{\cdot}(t)] + (1 + p)\mathcal{L}[x(t)] = \mathcal{L}\left[qe^{(-st-\tau)}\right] \qquad (4.7)$$

Using the properties of Laplace transforms, we have:

$$s^2X(s) - sx(0) - x^{\cdot}(0) + 2\zeta(sX(s) - x(0)) + (1 + p)X(s) = Q/(s + s') \qquad (4.8)$$

Where, $X(s)$is the Laplace transform of $x(t), x(0)$ is the initial value of $x(t), x^{\cdot}(0)$ is the initial value of $x^{\cdot}(t), Q$ is the Laplace transform of $qe^{(-s\tau)}$, and $s' = s + \zeta$. Rearranging the equation:

$$(s^2 + 2\zeta s + 1 + p)X(s) - sx(0) - x^{\cdot}(0) - 2\zeta x(0) = Q/(s + s') \qquad (4.9)$$

Now, let's consider the initial conditions. Assuming $x(0) = a$ and $x^{\cdot}(0) = b$, we can substitute these values into the equation:

$$(s^2 + 2\zeta s + 1 + p)X(s) - sa - b - 2\zeta a = q/s + s/(s + s) \qquad (4.10)$$

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Now, we can solve for X(s):

$$X(s) = [sx(0) + x^{\cdot}(0) - 2\zeta x(0) + Q/(s + s')] / [s^2 + 2\zeta s + (1 + p)] \quad (4.11)$$

let's rewrite the expression for $X(s)$ as:

$$X(s) = [sx(0) + x^{\cdot}(0) - 2\zeta x(0) + Q/(s + s')] / [s^2 + 2\zeta s + (1 + p)] \quad (4.12)$$

To decompose this expression into partial fractions, we assume that the denominator factors into linear factors. The decomposition has the form:

$$X(s) = A/(s - \alpha) + B/(s - \beta) + \ldots$$

Where, $A, B, \ldots$ are the coefficients to be determined, and $\alpha, \beta, \ldots$ are the roots of the denominator polynomial. To determine the coefficients $A, B, \ldots$, we need to perform the partial fraction decomposition. To do this, we need to get the roots of the denominator polynomial $s^2 + 2\zeta s + (1 + p)$.

The roots can be found by solving the quadratic equation:

$$s^2 + 2\zeta s + (1 + p) = 0 \quad (4.13)$$

let's denote the roots as α and β:

$$s^2 + 2\zeta s + (1 + p) = (s - \alpha)(s - \beta) = 0 \quad (4.14)$$

Once we have the roots, we can express the inverse Laplace transform of $X(s)$ in terms of these roots and the initial conditions $x(0)$ and $x^{\cdot}(0)$.

To determine the coefficients A and B, we can use the method of partial fractions. multiplying both sides of the equation by the denominator (s - α) (s - β), we have:

$[s^2 + 2\zeta s + (1 + p)]X(s) = (A/ (s - \alpha) + B/ (s - \beta)) * (s - \alpha) (s - \beta)$

Expanding and simplifying the right side, we get:

$s^2X(s) + 2\zeta sX(s) + (1 + p) X(s) = A (s - \beta) + B (s - \alpha)$

Now, we substitute the expression for $X(s)$ and rearrange the equation:

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$$[s^2 + 2\zeta s + (1 + p)](A/(s - \alpha) + B/(s - \beta)) = A(s - \beta) + B(s - \alpha) \qquad (4.15)$$

Next, we equate the coefficients of corresponding powers of s on both sides. The coefficient of s^2 on the left side is 1, and on the right side, it is $A + B$. The coefficient of s is 2ζ on the left side, and on the right side, it is $-A\alpha - B\beta$. The constant term on the left side is $(1 + p)$, and on the right side, it is $-A\beta - B\alpha$.

Setting up the equations based on the coefficients:

$$A + B = 1$$

$$-A\alpha - B\beta = 2\zeta$$

$$-A\beta - B\alpha = 1 + p \qquad (4.16)$$

Solving these equations will give us the values of A and B, which we can then use to compute the inverse Laplace transform.

So determine A and B, we need to find the values of α and β. These roots can be obtained by solving the quadratic equation:

$$s^2 + 2\zeta s + (1 + p) = 0 \qquad (4.17)$$

Using the quadratic formula, the roots α and β can be expressed as:

$$\alpha = (-2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))}) / 2$$

$$\beta = (-2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))}) / 2$$

Once we have the roots α and β, we can substitute them back into the equation for $X(s)$ and find the coefficients A and B using algebraic methods or simultaneous equations.

Substituting α and β into this equation, we have:

$$X(s) = A / (s - (-2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))}) / 2) + B / (s - (-2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))}) / 2) \qquad (4.18)$$

Simplifying the expressions:

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$$X(s) = A / (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p)) / 2)} + B / (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p)) / 2)} \quad (4.19)$$

Combining the fractions:

$$X(s) = (A * (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p)))} + B * (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))))} / (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p)))} * \left(s + 2\zeta + \sqrt{4\zeta^2 - 4(1 + p)}\right) \quad (4.20)$$

let's proceed with the partial fraction decomposition:

$$X(s) = (A * (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p)))} + B * (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))))} / ((s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p)))} * (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))))}$$

$$= A / (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p)))} + B / (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p)))} \quad (4.21)$$

Now, we can find the inverse Laplace transform of each term using standard Laplace transform tables or formulas. The inverse Laplace transform of $A / (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p)))}$ and $B / (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p)))}$ can be computed separately.

let's denote the inverse Laplace transform of $A / (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p)))}$ as $x_1(t)$ and the inverse Laplace transform of $B / (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p)))}$ as $x_2(t)$.

Once these inverse Laplace transforms are computed, the overall inverse Laplace transform of X(s) can be written as:

$$x(t) = x_1(t) + x_2(t) \quad (4.22)$$

To study the stability of the given differential equation using Laplace transforms, we need to analyze the poles of the transfer function obtained from the Laplace transform.

The transfer function is given by:

$$H(s) = X(s) / F(s) = [(sa + b + 2\zeta a) / [(s^2 + 2\zeta s + 1 + p) - q/s - s/(s + s)]] \quad (4.23)$$

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To study stability, we need to analyze the poles of the transfer function $X(s)$.

the transfer function $X(s)$ can be written as:

$$X(s) = (A * (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))}) + B * (s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))})) / ((s + 2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))}) * (s + 2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))})) \quad (4.24)$$

To assess stability, we need to examine the location of the poles of $X(s)$ in the complex plane. Stability is typically determined by the real parts of the poles.

let's denote the roots of the denominator as $s_1 = -2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))}$ and
$s_2 = -2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))}$.

To determine the stability of the system, we need to examine the real parts of the poles. let's denote the roots of the denominator as $s_1 = -2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))}$ and
$s_2 = -2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))}$.

For stability, we need both s_1 and s_2 to have negative real parts. This means that both $-2\zeta - \sqrt{(4\zeta^2 - 4(1 + p))}$ and $-2\zeta + \sqrt{(4\zeta^2 - 4(1 + p))}$ should be negative.

To simplify the stability analysis, let's focus on the expression inside the square root:

$$4\zeta^2 - 4(1 + p)$$

To calculate the value of $4\zeta^2 - 4(1 + p)$, we can follow these steps:

1. Start with the expression $4\zeta^2 - 4(1 + p)$.
2. Simplify the expression by performing the multiplication and addition/subtraction operations.
3. Substitute the specific values of ζ and p into the expression.
4. Evaluate the expression to obtain the numerical value.

let's go through an example to illustrate the calculation:

Suppose we have $\zeta = 0.5$ and $p = 2$. We can calculate $4\zeta^2 - 4(1 + p)$ as follows:

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$4\zeta^2 - 4(1 + p) = 4(0.5)^2 - 4(1 + 2)$

$= 4(0.25) - 4(3)$

$= 1 - 12$

$= -11$ (4.25)

So, in this example, the value of $4\zeta^2 - 4(1 + p)$ is -11.

By substituting the specific values of ζ and p into the expression and evaluating it, we can calculate the value of $4\zeta^2 - 4(1 + p)$ for particular case. This value will help to determine the stability of the system, we need to evaluate the value of $4\zeta^2 - 4(1 + p)$. If this value is negative or zero, both roots of the denominator will have negative real parts, indicating stability.

Therefore, if $4\zeta^2 - 4(1 + p)$ is negative or zero, it means that the system is stable. However, if $4\zeta^2 - 4(1 + p)$ is positive, the system will be unstable.

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