Study on the Existence of Solutions for Some Singular Nonlinear Equations on Exterior Domain

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ABSTRACT. This paper focuses on investigating the presence of solutions for singular non-linear equations (1.1) in the exterior domain. $R > 0$ on the boundary where $N > 2$. $f(u) \sim \frac{-1}{|u|^{q-1}u}$ for $|u|$ small and $0 < q < 1$, and $f(u) \sim |u|^{p-1}u$, for $|u|$ large and where $0 < p < 1$. also $K(x) = |x|^{-\alpha}$ with $\alpha > 2(N-1)$.

Keywords: Existence, singular, exterior, non-linear

INTRODUCTION

This paper involves the examination of the equation:

$$\Delta u + K(x)f(u) = 0, \quad \text{for } R < |x| < \infty \qquad (1.1)$$

$$u = 0 \quad on \quad R < |x| < \infty \qquad (1.2)$$

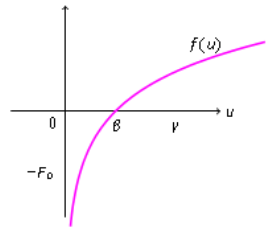
$$u \to 0 \ as \ |x| \to \infty \qquad (1.3)$$

Where the Laplacian operator is denoted by Δ, the function

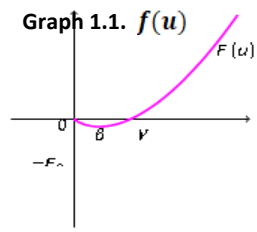
f: R\{0}→R, is locally Lipschitz. and : $\mathbb{R}^N \to \mathbb{R}$, $K(x) > 0$.

(H_1) f is odd, there exists $\beta > 0$ such that $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞) and $f(\beta) = 0$.

(H_2) $g_1 : R \to R$ *is* continuous and $f(u) = \frac{-1}{|u|^{q-1}u} + g_1(u)$ if u is small.



Graph 1.1. $f(u)$



Where $0 < q < 1$ and $g_1(0) = 0$.

(H_3) $g_2: R \to R$ is continuous and $f(u) = |u|^{p-1}u + g_2(u)$, if u is large. Where $0 < p < 1$. we let $F(u) = \int_0^u f(s)ds$. Since f is an odd function, it can be concludedthat F is an even function. Additionally, based on the assumption (H_2), it can be inferred that f is capable of being integrated into the vicinity of $u = 0$. This implies that F is a continuous function with a value of 0 at $u = 0$. Furthermore, F is bounded below by $-F_0$with $F_0 > 0$. Finally, from assumption (H_3), it can be determined that there exists a value of γ such that $0 < \beta < \gamma$.

Graph 1.2 . $F(u)$

(H_4) $F < 0$ on $(0, \gamma), F > 0$ on(γ, ∞), and $F > -F_0$ on $\mathbb{R}$, $F(\gamma) = 0$.

(H_5) The functions k and k' are characterized as having continuity on the interval $[R, \infty)$. It is also known that $k(r) > 0$, and that $k(r)$ is equal to $r - \alpha(2N - 1)$. Additionally, $2(N-1) + \frac{rk'}{k} < 0$. It is further assumed that α is greater than $2(N-1)$, $\lim_{r\to\infty} rk'/k = -\alpha$.

(H_6) There exists $k_1, k_2 > 0$ such that $\frac{k_1}{r^\alpha} \le k(r) \le \frac{k_2}{r^\alpha}$.

Lemma 1 : Assuming that the given conditions are true, with $f\big(u(r)\big)$ defined as $\frac{-1}{u^q} + u^p$, where $0 < \mathrm{p}, \mathrm{q} < 1$, and $u(r_\beta) = \beta$ under the assumption H_1, it can be deduced that the value of $\beta = 1$

Proof: Suppose $f\big(u(r)\big) = \frac{-1}{u^q} + u^p$, Given that both $0 < p, q < 1$, it is necessary to make an estimation of r at r_β. This estimation is made based on the fact that $u(r_\beta) = \beta$.

$$0 = f(\beta) = \frac{-1}{\beta^q} + \beta^p$$

We obtain :

$$-1 + \beta^{p+q} = 0$$

implies that:

$\beta^{p+q} = 1$ then by taking the root $(p+q)$ of both sides :

thus $\beta = 1$. The end of the prove lemma 1.

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Lemma 2: Let $F(u) = \frac{-u^{1-q}}{1-q} + \frac{u^{p+1}}{p+1}$ with $0 < \mathrm{p,q} < 1$, and $u(r_\gamma) = \gamma$ then the value $= (\frac{p+1}{1-q})^{\frac{1}{q+p}}$.

Proof: Suppose $F(u) = \frac{-u^{1-q}}{1-q} + \frac{u^{p+1}}{p+1}$, where $0 < \mathrm{p,q} < 1$ then we evaluate r at r_γ. We get :

$$0 = \frac{-\gamma^{1-q}}{1-q} + \frac{\gamma^{p+1}}{p+1}$$

$$0 = -(p+1)\gamma^{1-q} + (1-q)\gamma^{p+1}$$

Simplifying above we get:

$$\gamma^{p+q} = \frac{p+1}{1-q}$$

then by taking the root $(p+q)$of both sides. We obtain:

$$\gamma = \left(\frac{p+1}{1-q}\right)^{\frac{1}{p+q}}.$$

This completes the proof of lemma 2 .

Recent publications [1-10] regarding solutions of differential equations on the exterior domain piqued interest in the subject of this paper. this article is to study how the singular non-linear differential equation (1.1) behavior and to investigate the solutions that it produces. When $0 < p, q < 1$, and $\alpha > 2(N-1)$ are true. Additionally, in [1-10] the radial solutions were studied.

PRELIMINARIES

To study the existence of solutions for equations (1.1) through (1.3), it is necessary to make certain assumptions. Specifically, we assume that $r = |x|$, and that $u(r) = u(|x|)$, where $x \in R^N$.and satisfies

$$u''(r) + \frac{N-1}{r}u'(r) + K(|r|)f(u) = 0 \quad \text{On } (R, \infty) \tag{1.4}$$

$$\mathrm{u(R)} = 0 \qquad \lim_{r\to\infty} u(r) = 0 \tag{1.5}$$

to prove the existing solution we change the variable

$$u(r) = v_a(r^{2-N}) \tag{1.6}$$

Therefore

$$u'(r) = v_a{}'(r^{2-N})(2-N)r^{1-N} \tag{1.7}$$

$$u''(r) = (2-N)(1-N)r^{-N}v'(r^{2-N}) + (2-N)^2 r^{2(1-N)}\, v_a{}''(r^{2-N}) \tag{1.8}$$

Letting $t = r^{2-N}$ and $r = t^{\frac{1}{2-N}}$ in $(1.4) - (1.5)$ we obtain:

$$v_a{}'' + h(t)f(v_a) = 0 \qquad \text{on } (0, R^\tau) \tag{1.9}$$

$$v_a(R^\tau) = 0 \tag{1.10}$$

$$v_a{}'(R^\tau) = -a^\tau < 0 \tag{1.11}$$

where $R^{2-N} = R^\tau$ and $a^\tau = \frac{aR^{N-1}}{N-2}$

Then where form (H_4) and (H_5) and suppose :

$h(t) = \frac{t^{\hat{\alpha}}}{(N-2)^2}$ where $\hat{\alpha} = \frac{\alpha - 2(N-1)}{N-2} > 0$. Since $\alpha > 2(N-1)$, and $t \in (0, R^\tau)$. Also from (1.7) and (1.8), This suggests that there are constants h_1 and h_2, where $0 \le h_1 \le h_2$, which implies that...

$$h(R^\tau) \ge h(t) \tag{1.12}$$

$$h'(t) > 0, h_1 t^{\hat{\alpha}} \le h(t) \le h_2 t^{\hat{\alpha}} \quad \text{on } (0, R^\tau). \tag{1.13}$$

Theorem 1.1. Suppose that $a > 0$, $\alpha > 2(N-1)$, where $N > 2$ and $(H_1) - (H_6)$ hold. then there exists a solution of the equations (1.9)-(1.11) on $(R^\tau - \in, R^\tau)$ for some $\in > 0$.

Proof:

step (1):- Using the initial condition (1.10)-(1.11), we first integrate (1.9) on (t, R^τ), to obtain:

$$v_a(R^\tau) = 0, \; v_a'(R^\tau) = -a^\tau < 0$$

Integrating (1.9) on (t, R^τ) and using the initial condition we get:

$$-a^\tau - v_a'(t) = -\int_t^{R^\tau} h(t)f\big(v_a(s)\big)\, ds \tag{1.14}$$

$$v_a'(t) = -a^\tau + \int_t^{R^\tau} h(t)f\big(v_a(s)\big)\, ds \tag{1.15}$$

Also, integrating (1.15) on (t, R^τ) and with (1.11) we get :

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$$-v_a(t) = v_a(R^\tau) - v_a(t) = -a^\tau(R^\tau - t) + \int_t^{R^\tau} \int_s^{R^\tau} h(x) f\big(v_a(x)\big) dx ds$$

$$v_a(t) = a^\tau(R^\tau - t) - \int_t^{R^\tau} \int_s^{R^\tau} h(x) f\big(v_a(x)\big) dx ds \tag{1.16}$$

Now, we let $D^\tau(t) = \frac{v_a(t)}{(R^\tau - t)}$ (1.17)

And $v_a(t) = D^\tau(t)\big(R^\tau - t\big)$ and using L'Hôpital's rule and we obtain:

$$\lim_{t \to (R^\tau)^-} \frac{v_a(t)}{(R^\tau - t)} = -v_a(R^\tau) = a^\tau$$

And rewriting the (1.16) and we get :

$$D^\tau(t) = a^\tau - \frac{1}{(R^\tau - t)} \int_t^{R^\tau} \int_s^{R^\tau} h(x) f((R^\tau - x)\ D^\tau(x)) dx ds \tag{1.18}$$

Step(2): We will use the fixed point method to solve equation (1.18). To do so, we must establish a set using the fixed-point method. and let, $a > 0, 0 < \epsilon < 1$, and then define:

$$A = \left\{ D^\tau \in C[R^\tau - \epsilon, R^\tau] \,|\, |D^\tau(t) - a^\tau| \leq \frac{1}{2} a^\tau \text{on } [R^\tau - \epsilon, R^\tau] \right\}$$

Where $D^\tau(R^\tau) = a^\tau$ and $C[R^\tau - \epsilon,\ R^\tau]$ is the set of real–valued continuous functions on $[R^\tau - \epsilon,\ R^\tau]$. Let :

$$\|D^\tau\| = \sup_{x \in [R^\tau - \epsilon,\ R^\tau]} |D^\tau(x)|.$$

Then $(A, \|.\|)$ is a Banach space.

Now let us define a map φ on A by $\varphi D^\tau(R^\tau) = a^\tau$ and :

$$\varphi D^\tau(t) = D^\tau(t) = a^\tau - \frac{1}{(R^\tau - t)} \int_t^{R^\tau} \int_s^{R^\tau} h(x) f((R^\tau - x) D^\tau(x)) dx ds \tag{1.19}$$

$\varphi: A \to A$, $R^\tau - \epsilon < t < R^\tau$. since $D^\tau(x) \in A$ and $0 < \epsilon < 1$ we have:

$$\left|D^\tau(t) - a^\tau\right| \leq \frac{1}{2} a^\tau$$

$$-\frac{1}{2} a^\tau \leq D^\tau(x) - a^\tau \leq \frac{1}{2} a^\tau$$

$$a^\tau - \frac{1}{2}a^\tau \le D^\tau(x) \le \frac{1}{2}a^\tau + a^\tau$$

$$\frac{1}{2}a^\tau \le D^\tau \le \frac{3}{2}a^\tau \qquad \text{on } [R^\tau - \epsilon, R^\tau] \tag{1.20}$$

Now we estimate $f((R^\tau - x)D^\tau) = \frac{-1}{((R^\tau - x)^q D^{\tau q}} + g_1((R^\tau - x)D^\tau)$.

It follows form (1.20) that:

$$\left(\frac{1}{2}a^\tau \le D^\tau(x) \le \frac{3}{2}a^\tau\right)^q (R^\tau - x)^q$$

$$\frac{(R^\tau - x)^q}{2^q}a^{\tau q} \le D^{\tau q}(x)(R^\tau - x)^q \le \left(\frac{3}{2}\right)^q a^{\tau q}(R^\tau - x)^q$$

$$\frac{2^q}{(R^\tau - x)^q a^{\tau q}} \ge \frac{1}{D^{\tau q}(x)(R^\tau - x)^q} \ge \frac{1}{\left(\frac{3}{2}\right)^q a^{\tau q}(R^\tau - x)^q}$$

$$\frac{a^{\tau q}}{(R^\tau - x)^q 2^q} \ge (R^\tau - x)^{-q}(D^\tau(x))^{-q} \ge \frac{2^q a^{\tau q}}{(R^\tau - x)^q 3^q}$$

Now evaluate $\frac{-1}{(R^\tau - x)\, D^{\tau q}(x)}$

$$\left|\frac{-1}{(R^\tau - x)^q ((D^\tau(x))^q}\right| \le \frac{2^q}{a^{\tau q}}(R^\tau - x)^{-q}\,. \tag{1.21}$$

From H_2 we see $g_1(x)$ is locally Lipschitz, $0 < p < 1$, and $g_1(0) = 0$ therefore it follows that :

$$g_1((R^\tau - x)D^\tau(x)) \le L|R^\tau - x||D^\tau(x)| \tag{1.22}$$

$$g_1\big((R^\tau - x)D^\tau(x)\big) \le L|R^\tau - x|\ \frac{3}{2}a^\tau = \frac{3}{2}a^\tau\ L|R^\tau - x|$$

Where L is the Lipschitz constant for g_1 on $[0, \frac{3}{2}a^\tau]$.

Using (1.11),(1.21) and (1.22). Since $h' > 0$, then $h(t) \le h(R^\tau)$ and $0 < \mathrm{p}, \mathrm{q} < 1$ we see :

$$\left|h(x)f\big((R^\tau - x)D^\tau(x)\big)\right| = \left|h(x)\left(\frac{-1}{(R^\tau - x)^q((D^\tau(x))^q} + g_1\big(R^\tau - x\big)D^\tau(x)\right)\right|$$

$$\le h(R^\tau)\left[\left|\frac{2^q}{a^{\tau q}}\big(R^\tau - x\big)^{-q}\right| + L\left|(R^\tau - x)\frac{3}{2}a^\tau\right|\right. \tag{1.23}$$

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Integrating on (t, R^{τ}) and using (1.23) we get:

$$\int_t^{R^{\tau}} |h(x)f((R^{\tau}-x)D^{\tau}(x))|dx \le h(R^{\tau})\left[\frac{c_1}{a^{\tau q}}\,(R^{\tau}-t)^{1-q} + c_2 a^{\tau}(R^{\tau}-t)^2\right] \tag{1.24}$$

Where $c_1 = \frac{2^q}{a^{\tau q}}\frac{1}{1-q}$ and $c_2 = \frac{3LR^{N-1}}{4}$ .thus form (1.24)we see:

$$\int_t^{R^{\tau}} |h(x)f((R^{\tau}-x)D^{\tau}(x))|dx \to 0 \text{ as } t \to (R^{\tau})^{-} \ . \tag{1.25}$$

Next, divide by $(R^{\tau}-t)$ and integrate (1.24) on (t, R^{τ}). we obtain :

$$\frac{1}{R^{\tau}-t}\int_t^{R^{\tau}}\int_s^{R^{\tau}} \left|h(x)f\big((R^{\tau}-x)D^{\tau}(x)\big)\right|dxds \le h(R^{\tau})\left[\frac{c_3(R^{\tau}-t)^{1-q}}{a^{\tau q}} + a^{\tau}c_4(R^{\tau}-t)^2\right. \tag{1.26}$$

where $c_3 = \frac{c_1}{2-q}$ and $c_4 = \frac{c_2}{3}$ thus from(1.30)we see :

$$\lim_{t\to(R^{\tau})^{-}}\left[\frac{1}{R^{\tau}-t}\int_t^{R^{\tau}}\int_s^{R^{\tau}} \left|h(x)f\big((R^{\tau}-x)D^{\tau}(x)\big)\right|dxds\right] = 0. \tag{1.27}$$

Now show that $\varphi: A \to A$ is a contraction mapping with $\varphi(D^{\tau}) \in A$ for each $D \in A$ if $\epsilon > 0$ is sufficiently small.

First, let $D^{\tau} \in A$ and so it follows from (1.26)-(1.27) that:

$$\frac{1}{R^{\tau}-t}\int_t^{R^{\tau}}\int_s^{R^{\tau}} \left|h(x)f\big((R^{\tau}-x)D^{\tau}(x)\big)\right|dxds$$

Is continuous on $[R^{\tau}-\epsilon, R^{\tau}]$.then from(1.18),(1.26) and(1.27) we see: $\lim_{t\to(R^{\tau})^{-}} \varphi D^{\tau}(t) = a^{\tau}$,

$|\varphi D^{\tau}(t) - a^{\tau}| \le \frac{1}{2}a^{\tau}$ on $[R^{\tau}-\epsilon, R^{\tau}]$ and φD^{τ} continuous if $\epsilon > 0$ is sufficiently small.

Step(3): We need to prove that φ is a contraction mapping if ϵ is sufficiently small.

Let $D_1,\ D_2 \in A$ then:

$$\varphi D_1(t) - \varphi D_2(t) = \frac{-1}{R^{\tau}-t}\int_t^{R^{\tau}}\int_s^{R^{\tau}} h(x)\big[f\big((R^{\tau}-x)D_1(x)\big) - f\big((R^{\tau}-x)D_2(x)\big)\big]dxds \tag{1.28}$$

By (1.8) we have:

$$f\left((R^{\tau}-x)D^{\tau}(x)\right)=-(R^{\tau}-x)^{-q}D^{-q}(x)+g_1\left(R^{\tau}-x\right)D^{\tau}(x))$$

Where $0<p,q<1$.

Use (1.20) and (1.21).

We first estimate:

$$\begin{aligned}&\left|f\left((R^{\tau}-x)D_1\right)-f((R^{\tau}-x)D_2)\right|\\&\quad=\left|\frac{-1}{(R^{\tau}-x)^{q}}\left[\frac{1}{D_1^{q}}-\frac{1}{D_2^{q}}\right]+g_1\left((R^{\tau}-x)D_1\right)-g_1\left((R^{\tau}-x)D_2\right)\right|\\&\quad\le\frac{1}{(R^{\tau}-x)^{q}}\left|\frac{1}{D_1^{q}}-\frac{1}{D_2^{q}}\right|+L(R^{\tau}-x)|D_1-D_2|\end{aligned}\tag{1.29}$$

Where L is the Lipschitz constant for g_1 on $[0,\frac{3}{2}a^{\tau}]$.

Next applying the mean value theorem, we see that the right-hand side of (1.29) is equal to

$$\frac{1}{(R^{\tau}-x)^{q}}\left[\frac{q}{D_3^{q+1}}|D_1-D_2|\right]+L(R^{\tau}-x)|D_1-D_2|$$

where D_3 is between D_1 and D_2.

since $D_i\in S$ for $i=1,2,3$ and $|D_i-a^{\tau}|\le\frac{1}{2}a^{\tau}$ then $\frac{1}{2}a^{\tau}\le D_i\le\frac{3}{2}a^{\tau}$ on

$[R^{\tau}-\epsilon,\ R^{\tau}]$. Therefore it follows that $D_3^{q+1}\ge\left(\frac{1}{2}a^{\tau}\right)^{q+1}$ and so on$[R^{\tau}-\epsilon,R^{\tau}]$ we have:

$$\begin{aligned}&\left|f\left((R^{\tau}-x)D_1\right)-f\left((R^{\tau}-x)D_2\right)\right|\\&\quad\le|D_1-D_2|\left[\frac{q}{(R^{\tau}-x)^{q}}\left(\frac{2}{a^{\tau}}\right)^{q+1}+L(R^{\tau}-x)\right].\end{aligned}\tag{1.30}$$

From (1.12) $h(t)>0$ is continuous and increasing on $(0,R^{\tau}]$ with $\alpha>2(N-1)$ we see:

$$|\varphi D_1-\varphi D_2|\le\frac{h(R^{\tau})}{R^{\tau}-t}\int_t^{R^{\tau}}\int_s^{R^{\tau}}|D_1-D_2|\left[\frac{q2^{q+1}}{(R^{\tau}-x)^{q}\,(a^{\tau})^{q+1}}+L(R^{\tau}-x)\right]dxds$$

$$\le\frac{h(R^{\tau})}{R^{\tau}-t}\|D_1-D_2\|\int_t^{R^{\tau}}\int_s^{R^{\tau}}\left[\frac{q2^{q+1}}{(R^{\tau}-x)^{q}\,(a^{\tau})^{q+1}}+L(R^{\tau}-x)\right]dxds$$

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$$\leq h(R^{\tau})\|D_1 - D_2\|\left[\frac{c_5\epsilon^{1-q}}{a^{q+1}} + c_6\epsilon^2\right] = c_7\|D_1 - D_2\| \qquad (1.31)$$

Where. $c_5 = \frac{q}{(2-q)(1-q)}\left(\frac{2(N-2)}{R^{N-1}}\right)^{q+1}$, $c_6 = \frac{L}{6}$ and $c_7 = h(R^{2-N})\left[\frac{c_5\epsilon^{1-q}}{a^{q+1}} + c_6\epsilon^2\right]$ since:

$\lim_{\epsilon\to 0^+} c_7 = \lim_{\epsilon\to 0^+} h((R^{\tau})[\frac{c_5\epsilon^{1-q}}{a^{q+1}} + c_6\epsilon^2] = 0$ then for ϵ sufficiently small we see

$0 < c_7 < 1$ and therefore it follow (1.31) that φ is a contraction mapping Then by Contraction mapping principle on A we see there exists a unique solution. this complete proof of the theorem (1.1).

Now we define the energy equation of solution (1.9) –(1.11)

$$E_a(t) = \frac{1}{2}\frac{v_a'^2(t)}{h(t)} + F\big(v_a(t)\big) \qquad (1.32)$$

Lemma1.2. The energy equation $E_a(t)$ is non-increasing.

Proof: from the energy equation (1.32), Differentiating $E_a(t)$ and using (1.14) :

$$E_a'(t) = \frac{1}{2}\left[\frac{h(t)\big(2v_a'(t)\big)v_a''(t) - v_a'^2(t)\,h'(t)}{h^2(t)}\right] + f\big(v_a(t)\big)\,v_a'(t)$$

$$E_a'(t) = [\frac{v_a'(t)\,v_a''(t)}{h} + f\big(v_a(t)\big)\,v_a'(t) - \frac{v_a'^2(t)\,h'(t)}{2h^2(t)}]$$

$$E_a'(t) = \frac{v_a'(t)}{h(t)}[v_a''(t) + h(t)v_a'(t)] - \frac{v_a'^2(t)\,h'(t)}{2h^2(t)}$$

and since we know from(1.12) that $h'(t) > 0$ and $v_a'^2 > 0$ then:

$$E_a'(t) = -\frac{v_a'^2(t)\,h'(t)}{2h^2(t)} \leq 0 \qquad (1.33)$$

Thus E_a' is non – increasing where it is defined for this t with $t < R^{\tau}$ then :

$$0 < \frac{1}{2}\frac{a^{\tau}}{h(R^{\tau})} = E_a(R^{\tau}) \leq E_a(t) = \frac{1}{2}\frac{v_a^2(t)}{h(t)} + F\big(v_a(t)\big)\,. \qquad (1.33)$$

Thus $0 \leq E(t)$.

Applications

In this passage, we will illustrate the primary outcome attained in theorem (1.1) through the use of an illustration. Let us examine the subsequent differential non-linear equation:
$v_a''(t) + h(t)(-v_a^{-q} + v_a^{p}) = 0.$

with the initial condition

$v_a(R^\tau) = 0, v_a'(R^\tau) = -a^\tau,$ to find the existence of solutions

Let $p = \frac{1}{2}, q = -\frac{1}{2}$ let $a^\tau = 2$ and $R^\tau = 2,$ therefore:

$$v_a''(t) + h(2)\left(-v_a^{-\frac{1}{2}} + v_a^{\frac{1}{2}}\right) = 0\,, \qquad \text{on } (0,2). \tag{1.34}$$

With initial condition:

$$v_a(2) = 0,\ v_a'(2) = -2 \tag{1.35}$$

Integrate the equation (1.34) on (t,2), we get :

$$v_a'(2) - v_a'(t) = -\int_t^2 h(2)(-v_a^{-\frac{1}{2}} + v_a^{\frac{1}{2}})dt$$

$$-v_a'(t) = 2 - \int_t^2 h(2)(-v_a^{-\frac{1}{2}} + v_a^{\frac{1}{2}})dt \tag{1.36}$$

Integrate (1.36) on $(t, 2)$, we get :

$$v_a(t) = 2(2-t) - \int_t^2 \int_s^2 h(2)(-v_a^{-\frac{1}{2}} + v_a^{\frac{1}{2}})dsdt$$

And $D^\tau(t) = \frac{v_a(t)}{2-t}$ then $v_a(t) = (2-t)D^\tau(t)$

$\lim_{t\to 2} D^\tau(t) = \lim_{t\to 2} \frac{v_a(t)}{2-t}$ by L'Hôpital's rule then we get:

$$\lim_{t\to 2} \frac{v_a(t)}{2-t} = \frac{-2}{-1} = 2 = D^\tau(t)$$

let us define
$A = \{D^\tau(t) \in C[2-\epsilon, 2] || D^\tau(t) - 2| \le 1 \text{ on } (2-\epsilon, 2)\}$ then $-1 \le D^\tau(t) - 2 \le 1$ and $1 \le D^\tau(t) \le 3$. Let $\|D^\tau(t)\| = \sup_{x\in[2-\epsilon,2]} |D^\tau(t)|.$

We will establish a mapping φ: A→A. Furthermore, given that f is locally Lipschitz then $|g_1(v_a(t))|$ on (0,2) then we get:

$$|g_1(v_a(t))| \le L|v_a(t)| \text{ then } |f| \le L(2-t)D^\tau(t)$$

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Also $h(t) \leq h(2)$, since $h'(t) > 0$

$$|h(t)f| \leq h(2)L(2-t)D^{\tau}(t)$$

$$|h(t)f| \leq 3h(2)L(2-t) = C^{*}D^{\tau} \text{ ,where } 1 < C^{*} < 1$$

Let $D_1, D_2 \in A$ then $|\varphi D_1 - \varphi D_2| \leq C|D_1 - D_2|$ on $(2-\epsilon, 2)$ then we obtain $0 < C < 1$

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