Solve System of Nonlinear Fredholm Integro-Differential Equations of Second Kind by using Quintin B-Spline

Yusr Hamad Jassim[1], Borhan F. Jumaa[2] and Ghassan E. Arif[3]

[1,3]Department of Mathematics/ College of Education for Pure Sciences/ Tikrit University, Salah Al-din, Iraq.

[2]Computer Science Department, College of Computer Science and Information Technology, University of Kirkuk, Kirkuk, Iraq.

[1]yh230025pep@st.tu.edu.iq

[2]borhan_nissan@uokirkuk.edu.iq

[3]ghasanarif@tu.edu.iq

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[1,3]Department of Mathematics/ College of Education for Pure Sciences/ Tikrit University, Salah Al-din, Iraq.

[2]Computer Science Department, College of Computer Science and Information Technology, University of Kirkuk, Kirkuk, Iraq.

[1]yh230025pep@st.tu.edu.iq

[2]borhan_nissan@uokirkuk.edu.iq

[3]ghasanarif@tu.edu.iq

Abstract

This paper investigates the numerical solution of systems of nonlinear Fredholm integro-differential equations of the second kind using the Quintic B-spline method. These equations are significant in various scientific fields, including fluid dynamics, biological models, and chemical kinetics, due to their complexity and widespread applications. Traditional analytical solutions are often impractical; hence, efficient numerical methods are essential. We extend the use of Quintic B-splines, previously applied to other types of integro-differential equations, to these systems. The method is described in detail, including the formulation of the integro-differential equations, the construction of Quintic B-spline interpolants, and the application of LU matrix factorization to solve the resulting system of equations. We present three numerical examples to demonstrate the accuracy and efficiency of the proposed method, comparing theoretical and numerical results using the maximum absolute error and least square error norms. The results show that the Quintic B-spline method provides

a reliable and accurate approach for solving complex integro-differential equations.

Keywords: Quintic B-spline, Nonlinear Fredholm integro-differential equations, Numerical methods, Approximate solutions, LU matrix factorization, Fluid dynamics.

1. Introduction:

Mathematical modeling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations and stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations. These equations arise in many fields like fluid dynamics, biological models and chemical kinetics. Integro-differential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution [1]. In [3], some methods are showed for solving Integro-differential equations such as some methods are showed for solving

Integro-differential equations such as El-gendi's, Wollfe's and Galerkin methods.Recently, the first order linear Fredholm Integro-differential equation is solved by using rationalized Haar functions method [6]. In [2], [11] others methods can be seen to solve Integro-differential equation. In [10] use Quintic B-Spline Method for Solving Sharma Tasso Oliver Equation. In [12] On Solution of Fredholm Integro-differential Equations Using Composite Chebyshev Finite Difference Method. In [58] Cubic B-splines collocation method for a class of partial integro-differential equation. In [7] Use of Cubic B-Spline in Approximating Solutions of Boundary Value Problems. In [8] Exponentially Fitted Finite Difference Approximation for Singularly Perturbed Fredholm Integro-Differential Equation. In [9] Split-step quintic B-spline collocation methods for nonlinear Schrödinger Equations. In [4] Finite Difference with Quintic B-Splines

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for Solving a system of nonlinear Volterra Integro–Differential Equations of integer order.

In this paper examines the system of first and second–order multi–type Fredholm integro–differential equations of the second kind. The formulation below describes the unknown functions within and the derivative of the unknown function outside the integral sign. For $1,2,\ldots,m$

$$U_i^{(4)}(x)+\sum_{k=1}^{3}\mathcal{F}_{ik}(x)U_i^{(k)}(x)+\mathcal{F}_{i0}(x)U_i(x)=\mathcal{G}_i(x)+\mathcal{P}U_i(x) \quad (1.1)$$

where

$$\mathcal{P}U_i(x)=\sum_{j=1}^{m}\int_0^b \mathcal{K}_{ij}(x,t)\,\mathcal{V}(U_j(t))dt\ ,\ i=1,2,\ldots,m,\quad x\in[0,1]$$

With the initial conditions:

$$U_i^{(k)}(x_0)=U_{i0}^{k},\qquad k=0,1,2,3,\ i=1,2,\ldots,m \quad (1.2)$$

where $U_i(t)$ are unknown functions, $U^{(k)}{}_i(x)$ are the derivative of unknown function, Where $\mathcal{V}(U_j(t))$ is a nonlinear function of $U_j(t)$, the functions $\mathcal{G}_i(x),\mathcal{F}_{ik}(x)\ i=1,\ \ldots,m$ and kernels $\mathcal{K}_{ij}(x,t),\ 1\le i,j\le m$ These are real–valued functions defined on subsets of $\mathcal{R}^3\ and\ \mathcal{R}^1$, respectively..

2. Quintic B–Spline Interpolation:

The Quintic B–spline interpolation is a linear combination of the five order B–spline basis as follows:

$$QB(t)=\sum_{e=\ell-3}^{\ell+3}\mathcal{p}_e B\mathbb{S}_e^5(t),\ \ \ell=0,1,2,\ldots,m \quad (1.3)$$

where $\mathcal{p}_e$ are unknown real coefficients and $B\mathbb{S}_e^5(t)$ are five–order B–spline functions

then,

$$\frac{d}{dt}QB(t)=\sum_{e=\ell-3}^{\ell+3}\mathcal{p}_e\frac{d}{dt}B\mathbb{S}_e^5(t),\ \ \ell=0,1,2,\ldots,m \quad (1.4)$$

and

$$\frac{d^2}{dt^2}QB(t)=\sum_{e=\ell-3}^{\ell+3}\mathcal{p}_e\frac{d^2}{dt^2}B\mathbb{S}_e^5(t),\ \ \ell=0,1,2,\ldots,m \quad (1.5)$$

and

$$\frac{d^3}{dt^3}QB(t) = \sum_{e=\ell-3}^{\ell+3} p_e \frac{d^3}{dt^3} B\mathbb{S}_e^5(t), \quad \ell = 0,1,2,\ldots,m \tag{1.6}$$

$$\frac{d^4}{dt^4}QB(t) = \sum_{e=\ell-3}^{\ell+3} p_e \frac{d^4}{dt^4} B\mathbb{S}_e^5(t), \quad \ell = 0,1,2,\ldots,m$$

According to the property of the quintic B−spline (1.3), can be simplified to

$$QB(t) = p_{\ell-3}B\mathbb{S}_{\ell-3}^5(t_\ell) + p_{\ell-2}B\mathbb{S}_{\ell-2}^5(t_\ell) + p_{\ell-1}B\mathbb{S}_{\ell-1}^5(t_\ell) + p_\ell B\mathbb{S}_\ell^5(t_\ell) + p_{\ell+1}B\mathbb{S}_{\ell+1}^5(t_\ell) + p_{\ell+2}B\mathbb{S}_{\ell+2}^5(t_\ell) + p_{\ell+3}B\mathbb{S}_{\ell+3}^5(t_\ell) \tag{1.7}$$

By shifting the quintic B−spline to the right side by m's step, mathematically meaning:

$$B\mathbb{S}_{\ell-m}^5(t_\ell)B\mathbb{S}_\ell^5(t_{\ell+m})$$

Then can rewrite the equation (1.7) as

$$QB(t) = p_{\ell-3}B\mathbb{S}_\ell^5(t_{\ell+3}) + p_{\ell-2}B\mathbb{S}_\ell^5(t_{\ell+2}) + p_{\ell-1}B\mathbb{S}_\ell^5(t_{\ell+1}) + p_\ell B\mathbb{S}_\ell^5(t_\ell) + p_{\ell+1}B\mathbb{S}_\ell^5(t_{\ell-1}) + p_{\ell+2}B\mathbb{S}_\ell^5(t_{\ell-2}) + p_{\ell+3}B\mathbb{S}_\ell^5(t_{\ell-3}) \tag{1.8}$$

Similarly, we can rewrite the equation (1.4), (1.5), (1.6) as following

$$\frac{d}{dt}QB(t) = p_{\ell-3}\frac{d}{dt}B\mathbb{S}_\ell^5(t_{\ell+3}) + p_{\ell-2}\frac{d}{dt}B\mathbb{S}_\ell^5(t_{\ell+2}) + p_{\ell-1}\frac{d}{dt}B\mathbb{S}_\ell^5(t_{\ell+1}) + p_\ell\frac{d}{dt}B\mathbb{S}_\ell^5(t_\ell) + p_{\ell+1}\frac{d}{dt}B\mathbb{S}_\ell^5(t_{\ell-1}) + p_{\ell+2}\frac{d}{dt}B\mathbb{S}_\ell^5(t_{\ell-2}) + p_{\ell+3}\frac{d}{dt}B\mathbb{S}_\ell^5(t_{\ell-3}) \tag{1.9}$$

$$\frac{d^2}{dt^2}QB(t) = p_{\ell-3}\frac{d^2}{dt^2}B\mathbb{S}_\ell^5(t_{\ell+3}) + p_{\ell-2}\frac{d^2}{dt^2}B\mathbb{S}_\ell^5(t_{\ell+2}) + p_{\ell-1}\frac{d^2}{dt^2}B\mathbb{S}_\ell^5(t_{\ell+1}) + p_\ell\frac{d^2}{dt^2}B\mathbb{S}_\ell^5(t_\ell) + p_{\ell+1}\frac{d^2}{dt^2}B\mathbb{S}_\ell^5(t_{\ell-1}) + p_{\ell+2}\frac{d^2}{dt^2}B\mathbb{S}_\ell^5(t_{\ell-2}) + p_{\ell+3}\frac{d^2}{dt^2}B\mathbb{S}_\ell^5(t_{\ell-3}) \tag{1.10}$$

$$\frac{d^3}{dt^3}QB(t) = p_{\ell-3}\frac{d^3}{dt^3}B\mathbb{S}_\ell^5(t_{\ell+3}) + p_{\ell-2}\frac{d^3}{dt^3}B\mathbb{S}_\ell^5(t_{\ell+2}) + p_{\ell-1}\frac{d^3}{dt^3}B\mathbb{S}_\ell^5(t_{\ell+1}) + p_\ell\frac{d^3}{dt^3}B\mathbb{S}_\ell^5(t_\ell) + p_{\ell+1}\frac{d^3}{dt^3}B\mathbb{S}_\ell^5(t_{\ell-1}) + p_{\ell+2}\frac{d^3}{dt^3}B\mathbb{S}_\ell^5(t_{\ell-2}) + p_{\ell+3}\frac{d^3}{dt^3}B\mathbb{S}_\ell^5(t_{\ell-3}) \tag{1.11}$$

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$$\frac{d^4}{dt^4}QB(t) = \mathcal{p}_{\ell-3}\frac{d^4}{dt^4}B\mathbb{S}_\ell^5(t_{\ell+3}) + \mathcal{p}_{\ell-2}\frac{d^4}{dt^4}B\mathbb{S}_\ell^5(t_{\ell+2}) + \mathcal{p}_{\ell-1}\frac{d^4}{dt^4}B\mathbb{S}_\ell^5(t_{\ell+1}) + \mathcal{p}_\ell\frac{d^4}{dt^4}B\mathbb{S}_\ell^5(t_\ell) + \mathcal{p}_{\ell+1}\frac{d^4}{dt^4}B\mathbb{S}_\ell^5(t_{\ell-1}) + \mathcal{p}_{\ell+2}\frac{d^4}{dt^4}B\mathbb{S}_\ell^5(t_{\ell-2}) + \mathcal{p}_{\ell+3}\frac{d^4}{dt^4}B\mathbb{S}_\ell^5(t_{\ell-3})$$

(1.12)

The following equations are formulated, by substituting the value of $B\mathbb{S}_\ell^5(t)$ at the Knots from table (1.1):

$$QB(t) = \mathcal{p}_{\ell-2} + 26\mathcal{p}_{\ell-1} + 66\mathcal{p}_\ell + 26\mathcal{p}_{\ell+1} + \mathcal{p}_{\ell+2} \quad (1.13)$$

$$\frac{d}{dt}QB(t) = -\frac{5}{h}\mathcal{p}_{\ell-2} - \frac{55}{h}\mathcal{p}_{\ell-1} + \frac{55}{h}\mathcal{p}_{\ell+1} + \frac{5}{h}\mathcal{p}_{\ell+2} \quad (1.14)$$

$$\frac{d^2}{dt^2}QB(t) = \frac{20}{h^2}\mathcal{p}_{\ell-2} + \frac{40}{h^2}\mathcal{p}_{\ell-1} - \frac{120}{h^2}\mathcal{p}_\ell + \frac{40}{h^2}\mathcal{p}_{\ell+1} + \frac{20}{h^2}\mathcal{p}_{\ell+2} \quad (1.15)$$

$$\frac{d^3}{dt^3}QB(t) = -\frac{60}{h^3}\mathcal{p}_{\ell-2} + \frac{120}{h^3}\mathcal{p}_{\ell-1} - \frac{120}{h^3}\mathcal{p}_{\ell+1} + \frac{60}{h^3}\mathcal{p}_{\ell+2} \quad (1.16)$$

$$\frac{d^4}{dt^4}QB(t) = \frac{120}{h^4}\mathcal{p}_{\ell-2} - \frac{480}{h^4}\mathcal{p}_{\ell-1} + \frac{720}{h^4}\mathcal{p}_\ell - \frac{480}{h^4}\mathcal{p}_{\ell+1} + \frac{120}{h^4}\mathcal{p}_{\ell+2} \quad (1.17)$$

3. Description of the Method:

In this section, we study the use of Quintic B−splines to solve the system of first and second−order multi−type Fredholm integro−differential equations of the second kind.

$$\frac{d^4}{dt^4}U_i(t) + \mathcal{F}_{i3}(t)\frac{d^3}{dt^3}U_i(t) + \mathcal{F}_{i2}(t)\frac{d^2}{dt^2}U_i(t) + \mathcal{F}_{i1}(t)\frac{d}{dt}U_i(t) + \mathcal{F}_{i0}(t)U_i(t) = \mathcal{G}_i(t) + \mathcal{P}U_i(t), \quad i = 1,2,\dots,m \quad (1.18)$$

where

$$\mathcal{P}U_i(t) = \sum_{j=1}^{m}\int_0^b \mathcal{K}_{ij}(t,x)\mathcal{V}(U_j(x))dt, \ i = 1,2,\dots,m, \ t \in [0,1]$$

With the initial conditions:

$$\left.\begin{aligned}\frac{d^4}{dt^4}U_i(t_0) &= U_{i0}^4\\ \frac{d^3}{dt^3}U_i(t_0) &= U_{i0}^3\\ \frac{d^2}{dt^2}U_i(t_0) &= U_{i0}^2\\ \frac{d}{dt}U_i(t_0) &= U_{i0}^1\\ U_i(t_0) &= U_{i0}\end{aligned}\right\}$$

(1.19)

where $U_i(t)$ are unknown functions, $\frac{d^4}{dt^4}U_i(t)$, $\frac{d^3}{dt^3}U_i(t)$, $\frac{d^2}{dt^2}U_i(t)$, $\frac{d}{dt}U_i(t)$ are the derivative of unknown function, Where $\mathcal{V}(U_j(x))$ is a nonlinear function of $U_j(x)$, the functions $\mathcal{G}_i(x)$, $i = 1, \ \dots, m$, $\mathcal{F}_{i3}(x)$, $\mathcal{F}_{i2}(x)$, $\mathcal{F}_{i1}(x)$, $\mathcal{F}_{i0}(x)$ and kernels $\mathcal{K}_{ij}(t,x)$, $1 \le i,j \le m$ These are real−valued functions defined on subsets of $\mathcal{R}^3$ and $\mathcal{R}^1$, respectively.

We assume equally spaced knots, i.e., $h = t_{\ell+1} - t_\ell$. Let

$$U_i(t) = QB_i(t) = \mathcal{p}_{i\ell-2} + 26\mathcal{p}_{i\ell-1} + 66\mathcal{p}_{i\ell} + 26\mathcal{p}_{i\ell+1} + \mathcal{p}_{i\ell+2} \tag{1.20}$$

$$\frac{d}{dt}U_i(t) = \frac{d}{dt}QB_i(t) = -\frac{5}{h}\mathcal{p}_{i\ell-2} - \frac{55}{h}\mathcal{p}_{i\ell-1} + \frac{55}{h}\mathcal{p}_{i\ell+1} + \frac{5}{h}\mathcal{p}_{i\ell+2} \tag{1.21}$$

$$\frac{d^2}{dt^2}U_i(t) = \frac{d^2}{dt^2}QB_i(t) = \frac{20}{h^2}\mathcal{p}_{i\ell-2} + \frac{40}{h^2}\mathcal{p}_{i\ell-1} - \frac{120}{h^2}\mathcal{p}_{i\ell} + \frac{40}{h^2}\mathcal{p}_{i\ell+1} + \frac{20}{h^2}\mathcal{p}_{i\ell+2} \tag{1.22}$$

$$\frac{d^3}{dt^3}U_i(t) = \frac{d^3}{dt^3}QB_i(t) = -\frac{60}{h^3}\mathcal{p}_{i\ell-2} + \frac{120}{h^3}\mathcal{p}_{i\ell-1} - \frac{120}{h^3}\mathcal{p}_{i\ell+1} + \frac{60}{h^3}\mathcal{p}_{i\ell+2} \tag{1.23}$$

$$\frac{d^4}{dt^4}U_i(t) = \frac{d^4}{dt^4}QB_i(t)) = \frac{120}{h^4}\mathcal{p}_{i\ell-2} - \frac{480}{h^4}\mathcal{p}_{i\ell-1} + \frac{720}{h^4}\mathcal{p}_{i\ell} - \frac{480}{h^4}\mathcal{p}_{i\ell+1} + \frac{120}{h^4}\mathcal{p}_{i\ell+2}$$

(1.24)

be the approximating function, It is required that (1.20)−(1.24) satisfies our the system of first and second−order multi−type Fredholm integro−differential equations of the second kind (1.18) and (1.19) at $t = t_i$.

where t_i is an interior point. That is

$$\frac{d^4}{dt^4}QB_i(t) + \mathcal{F}_{i3}(t)\frac{d^3}{dt^3}QB_i(t) + \mathcal{F}_{i2}(t)\frac{d^2}{dt^2}QB_i(t) + \mathcal{F}_{i1}(t)\frac{d}{dt}QB_i(t) + \mathcal{F}_{i0}(t)QB_i(t) = \mathcal{G}_i(t) + \sum_{j=1}^{m}\int_0^b \mathcal{K}_{ij}(t,x)\mathcal{V}(QB_j(x))dt \tag{1.25}$$

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i.e.

$$\frac{d^4}{dt^4}QB_1(t)+\mathcal{F}_{13}(t)\frac{d^3}{dt^3}QB_1(t)+\mathcal{F}_{12}(t)\frac{d^2}{dt^2}QB_1(t)+\mathcal{F}_{11}(t)\frac{d}{dt}QB_1(t)+ \mathcal{F}_{10}(t)QB_1(t)=\mathcal{G}_1(t)+\sum_{j=1}^{m}\int_0^b \mathcal{K}_{1j}(t,x)\mathcal{V}(QB_j(x))dt \qquad (1.26)$$

$$\frac{d^4}{dt^4}QB_2(t)+\mathcal{F}_{23}(t)\frac{d^3}{dt^3}QB_2(t)+\mathcal{F}_{22}(t)\frac{d^2}{dt^2}QB_2(t)+\mathcal{F}_{21}(t)\frac{d}{dt}QB_2(t)+ \mathcal{F}_{20}(t)QB_2(t)=\mathcal{G}_2(t)+\sum_{j=1}^{m}\int_0^b \mathcal{K}_{2j}(t,x)\mathcal{V}(QB_j(x))dt \qquad (1.27)$$

$$\vdots$$

$$\frac{d^4}{dt^4}QB_n(t)+\mathcal{F}_{n3}(t)\frac{d^3}{dt^3}QB_n(t)+\mathcal{F}_{n2}(t)\frac{d^2}{dt^2}QB_n(t)+\mathcal{F}_{n1}(t)\frac{d}{dt}QB_n(t)+ \mathcal{F}_{n0}(t)QB_n(t)=\mathcal{G}_n(t)+\sum_{j=1}^{m}\int_0^b \mathcal{K}_{nj}(t,x)\mathcal{V}(QB_j(x))dt \qquad (1.28)$$

B using the initial conditions (1.19), we get the following

$$\mathcal{P}_{i\ell-2}+26\mathcal{P}_{i\ell-1}+66\mathcal{P}_{i\ell}+26\mathcal{P}_{i\ell+1}+\mathcal{P}_{i\ell+2}=U_{i0}$$

$$-\frac{5}{h}\mathcal{P}_{i\ell-2}-\frac{55}{h}\mathcal{P}_{i\ell-1}+\frac{55}{h}\mathcal{P}_{i\ell+1}+\frac{5}{h}\mathcal{P}_{i\ell+2}=U_{i0}^1$$

$$\frac{20}{h^2}\mathcal{P}_{i\ell-2}+\frac{40}{h^2}\mathcal{P}_{i\ell-1}-\frac{120}{h^2}\mathcal{P}_{i\ell}+\frac{40}{h^2}\mathcal{P}_{i\ell+1}+\frac{20}{h^2}\mathcal{P}_{i\ell+2}=U_{i0}^2$$

$$-\frac{60}{h^3}\mathcal{P}_{i\ell-2}+\frac{120}{h^3}\mathcal{P}_{i\ell-1}-\frac{120}{h^3}\mathcal{P}_{i\ell+1}+\frac{60}{h^3}\mathcal{P}_{i\ell+2}=U_{i0}^3$$

$$\frac{120}{h^4}\mathcal{P}_{i\ell-2}-\frac{480}{h^4}\mathcal{P}_{i\ell-1}+\frac{720}{h^4}\mathcal{P}_{i\ell}-\frac{480}{h^4}\mathcal{P}_{i\ell+1}+\frac{120}{h^4}\mathcal{P}_{i\ell+2}=U_{i0}^4$$

We can write the above system as the matrix

$$AP=U$$

(1.29)

Where

$$A=\begin{bmatrix} 1 & 26 & 66 & 26 & 1 \\ -5 & -55 & 0 & 55 & 5 \\ 20 & 40 & -120 & 40 & 20 \\ -60 & 120 & 0 & -120 & 60 \\ 120 & -480 & 720 & -480 & 120 \end{bmatrix} \quad P=\begin{bmatrix} \mathcal{P}_{i\ell-2} \\ \mathcal{P}_{i\ell-1} \\ \mathcal{P}_{i\ell} \\ \mathcal{P}_{i\ell+1} \\ \mathcal{P}_{i\ell+2} \end{bmatrix} \quad U=\begin{bmatrix} U_{i0} \\ hU_{i0}^1 \\ h^2U_{i0}^2 \\ h^3U_{i0}^3 \\ h^4U_{i0}^4 \end{bmatrix}$$

And use LU matrix factorization to solve this system to evaluate the value of the coefficients $\mathcal{P}_{i\ell-2}, \mathcal{P}_{i\ell-1}, \mathcal{P}_{i\ell}, \mathcal{P}_{i\ell+1}, \mathcal{P}_{i\ell+2}$ at $t=t_0=0$

And use the equations (1.25)−(1.28) to find the value of the coefficients $\mathcal{p}_{i\ell-1}$, $\mathcal{p}_{i\ell}$, $\mathcal{p}_{i\ell+1}$, $\mathcal{p}_{i\ell+2}$ at $t = t_1, t_2, \ldots, t_n$.
and the approximate solution $U_i, i = 1,2,3,\ldots, n$ can be found from

$$U_i(t) = \sum_{e=-3}^{+3} \mathcal{p}_e B\mathbb{S}_e^5(t) \ .$$

4. Numerical Methods

In this section, we address three examples of the second type of nonlinear system of multi−type Fredholm integro−differential equations using a Quintic B−splines method. To analyze the numerical performance of the given method, we use two error measurements, that is, the maximum
absolute error L_∞ and $L.S.E.$ error norm which are defined by

$$L_\infty = \|U(exact) - U(approximate)\|_\infty$$
$$= \max_i |U_i(exact) - U_i(approximate)|$$

$$L.S.E. = \sum_{i=1}^{m} (U_i(exact) - U_i(approximate))^2$$

Example (1):

$$\left.\begin{aligned} u'(t) + 3u(t) &= 3t^2 - \frac{31}{15}t - \frac{11}{6} + \int_0^1 (t+y)\,[u^2(y) + v^2(y)]dy \\ v'(t) + 2v(t) &= \frac{32}{15}t^2 + 4t + \frac{11}{12} + \int_0^1 (t^2 - y)\,[u(y)v(y)]dy \end{aligned}\right\} \quad (1.30)$$

where $\quad \mathrm{u}(0) = 0, \mathrm{v}(0) = 0 \quad$ (1.31)

The exact solution $\quad u(t) = t^2 - t \quad$ and $\quad v(x) = t^2 + t$

Suppose that

$$u(t) = QB_1(t) = \mathcal{p}_{1,\ell-2} + 26\mathcal{p}_{1,\ell-1} + 66\mathcal{p}_{1,\ell} + 26\mathcal{p}_{1,\ell+1} + \mathcal{p}_{1,\ell+2} \quad (1.32)$$

$$\frac{d}{dt}u(t) = \frac{d}{dt}QB_1(t) = -\frac{5}{h}\mathcal{p}_{1,\ell-2} - \frac{55}{h}\mathcal{p}_{1,\ell-1} + \frac{55}{h}\mathcal{p}_{1,\ell+1} + \frac{5}{h}\mathcal{p}_{1,\ell+2} \quad (1.33)$$

$$v(t) = QB_2(t) = \mathcal{p}_{2,\ell-2} + 26\mathcal{p}_{2,\ell-1} + 66\mathcal{p}_{2,\ell} + 26\mathcal{p}_{2,\ell+1} + \mathcal{p}_{2,\ell+2} \quad (1.34)$$

$$\frac{d}{dt}v(t) = \frac{d}{dt}QB_2(t) = -\frac{5}{h}\mathcal{p}_{2,\ell-2} - \frac{55}{h}\mathcal{p}_{2,\ell-1} + \frac{55}{h}\mathcal{p}_{2,\ell+1} + \frac{5}{h}\mathcal{p}_{2,\ell+2} \quad (1.35)$$

$$u^2(t) = (QB_1(t))^2 = \left(\mathcal{p}_{1,\ell-2} + 26\mathcal{p}_{1,\ell-1} + 66\mathcal{p}_{1,\ell} + 26\mathcal{p}_{1,\ell+1} + \mathcal{p}_{1,\ell+2}\right)^2 \quad (1.36)$$

$$v^2(t) = (QB_2(t))^2 = \left(\mathcal{p}_{2,\ell-2} + 26\mathcal{p}_{2,\ell-1} + 66\mathcal{p}_{2,\ell} + 26\mathcal{p}_{2,\ell+1} + \mathcal{p}_{2,\ell+2}\right)^2 \quad (1.37)$$

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And $\int_0^1 (t+y)\,dy = t+\frac{1}{2}$, $\int_0^1 (t^2-y)\,dy = t^2-\frac{1}{2}$

Substation the equations (1.32) −(1.37) in to system (1.30), we obtain

$$-\frac{5}{h}p_{1,\ell-2}-\frac{55}{h}p_{1,\ell-1}+\frac{55}{h}p_{1,\ell+1}+\frac{5}{h}p_{1,\ell+2}+3(p_{1,\ell-2}+26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2})=3t^2-\frac{31}{15}t-\frac{11}{6}+\left(t+\frac{1}{2}\right)\left(\left(p_{1,\ell-2}+26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2}\right)^2\left(p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2}\right)^2\right)$$

(1.38)

$$-\frac{5}{h}p_{2,\ell-2}-\frac{55}{h}p_{2,\ell-1}+\frac{55}{h}p_{2,\ell+1}+\frac{5}{h}p_{2,\ell+2}+2(p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2})=\frac{32}{15}t^2+4t+\frac{11}{12}+(t^2-\frac{1}{2})(p_{1,\ell-2}+26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2})(p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2})$$

(1.39)

We rewrite the equations (1.38),(1.39) as the system of matrix, and use LU matrix factorization to solve this system to evaluate the value of the coefficients $p_{i\ell-2}, p_{i\ell-1}, p_{i\ell}, p_{i\ell+1}, p_{i\ell+2}$ at $t=t_0, t_1, \dots, t_{10}$, where $i=1,2$ and the approximate solution $u(t)$ and $v(t)$ can be found from

$$u(t)=\sum_{e=-3}^{+3} p_{1,e} B\mathbb{S}_e^5(t) \quad \text{and} \quad v(t)=\sum_{e=-3}^{+3} p_{2,e} B\mathbb{S}_e^5(t)$$

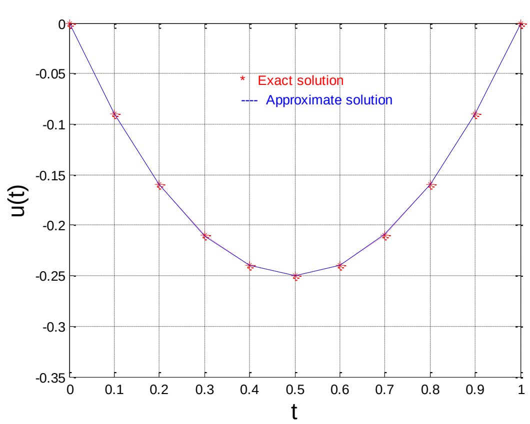
Table (1.1) displays a comparison between the theoretical and numerical results obtained using QBS for $u(t)$ in example 1, based on the least square error with h=0.1.

T	**Exact**	**QBS**	**Error(L_∞)**
0.000000000000000	**0.000000000000000**	**0.000000000000000**	**0.000000000000000**
0.100000000000000	**-0.090000000000000**	**-0.090132114993877**	**0.000132114993877**
0.200000000000000	**-0.160000000000000**	**-0.159431408530056**	**0.000568591469944**
0.300000000000000	**-0.210000000000000**	**-0.210800614669086**	**0.000800614669086**
0.400000000000000	**-0.240000000000000**	**-0.239746786881128**	**0.000253213118872**
0.500000000000000	**-0.250000000000000**	**-0.250226656277407**	**0.000226656277407**
0.600000000000000	**-0.240000000000000**	**-0.239622362429794**	**0.000377637570206**
0.700000000000000	**-0.210000000000000**	**-0.209066242146599**	**0.000933757853401**

0.800000000000000	-0.160000000000000	-0.159176289356898	0.000823710643102
0.900000000000000	-0.090000000000000	-0.090255344092533	0.000255344092533
1.000000000000000	0.000000000000000	-0.000418721162904	0.000418721162904
L.S.E	3.030765535946593e-006		

Table (1.2) displays a comparison between the theoretical and numerical results obtained using QBS for $v(t)$ in example 1, based on the least square error with h=0.1.

T	Exact	QBS	Error(L_∞)
0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000
0.100000000000000	0.110000000000000	0.110808583826749	0.000808583826749
0.200000000000000	0.240000000000000	0.239708009140380	0.000291990859620
0.300000000000000	0.390000000000000	0.390990917371084	0.000990917371084
0.400000000000000	0.560000000000000	0.560889732752114	0.000889732752114
0.500000000000000	0.750000000000000	0.749860377538571	0.000139622461429
0.600000000000000	0.960000000000000	0.959888259874967	0.000111740125033
0.700000000000000	1.190000000000000	1.189601614041920	0.000398385958080
0.800000000000000	1.440000000000000	1.439436750659063	0.000563249340937
0.900000000000000	1.710000000000000	1.709732509734401	0.000267490265599
1.000000000000000	2.000000000000000	1.999991161087142	0.000008838912858
L.S.E.	3.092178720991963e-006		



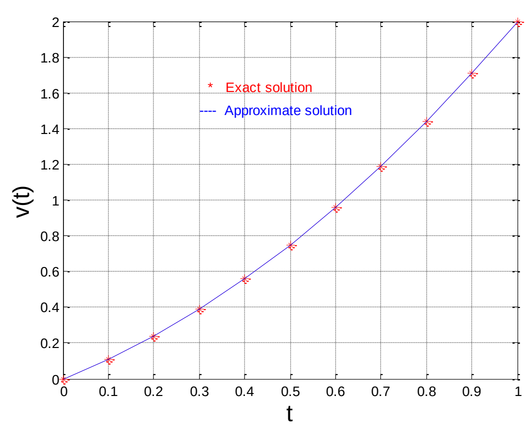


Fig.(1.1) displays a comparison between the exact and numerical solutions obtained using QBS for $u(t)$ and $v(t)$ in example 1, based on the least square error with h=0.1

Example (2)

Consider the following system of multi−type three nonlinear Fredholm integro−differential equations of the second kind.

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$$\left.\begin{aligned} u'(t)+t^2u(t)&=(t^2+1)e^t-e^{(t+1)}+\int_0^1\left[e^{(t-3y)}v^2(y)+e^{(t-6y)}w^2(y)\right]dy\\ v'(t)+tv(t)&=(t+2)e^{2t}-2e^t+\int_0^1\left[e^{(t-6y)}w^2(y)+e^{(t-2y)}u^2(y)\right]dy\\ w'(t)+2tw(t)&=(2t+3)e^{3t}-2e^t+\int_0^1\left[e^{(t-2y)}u^2(y)+e^{(t-4y)}v^2(y)\right]dy \end{aligned}\right\} \quad (1.40)$$

where $u(0)=1, v(0)=1, w(0)=1$ (1.41)

The exact solution are $u(t)=e^t$, $v(t)=e^{2t}$ and $w(t)=e^{3t}$

Suppose that

$$u(t)=QB_1(t)=p_{1,\ell-2}+26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2} \quad (1.42)$$

$$\frac{d}{dt}u(t)=\frac{d}{dt}QB_1(t)=-\frac{5}{h}p_{1,\ell-2}-\frac{55}{h}p_{1,\ell-1}+\frac{55}{h}p_{1,\ell+1}+\frac{5}{h}p_{1,\ell+2} \quad (1.43)$$

$$v(t)=QB_2(t)=p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2} \quad (1.44)$$

$$\frac{d}{dt}v(t)=\frac{d}{dt}QB_2(t)=-\frac{5}{h}p_{2,\ell-2}-\frac{55}{h}p_{2,\ell-1}+\frac{55}{h}p_{2,\ell+1}+\frac{5}{h}p_{2,\ell+2} \quad (1.45)$$

$$w(t)=QB_3(t)=p_{3,\ell-2}+26p_{3,\ell-1}+66p_{3,\ell}+26p_{3,\ell+1}+p_{3,\ell+2} \quad (1.46)$$

$$\frac{d}{dt}w(t)=\frac{d}{dt}QB_3(t)=-\frac{5}{h}p_{3,\ell-2}-\frac{55}{h}p_{3,\ell-1}+\frac{55}{h}p_{3,\ell+1}+\frac{5}{h}p_{3,\ell+2} \quad (1.47)$$

$$u^2(t)=(QB_1(t))^2=\left(p_{1,\ell-2}+26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2}\right)^2 \quad (1.48)$$

$$v^2(t)=(QB_2(t))^2=\left(p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2}\right)^2 \quad (1.49)$$

$$w^2(t)=(QB_3(t))^2=\left(p_{3,\ell-2}+26p_{3,\ell-1}+66p_{3,\ell}+26p_{3,\ell+1}+p_{3,\ell+2}\right)^2 \quad (1.50)$$

And $\int_0^1 e^{(t-2y)}\,dy=\frac{1}{2}(e^t-e^{t-2})$, $\int_0^1 e^{(t-3y)}\,dy=\frac{1}{3}(e^t-e^{t-3})$,
$\int_0^1 e^{(t-4y)}\,dy=\frac{1}{4}(e^t-e^{t-4})$, and $\int_0^1 e^{(t-6y)}\,dy=\frac{1}{6}(e^t-e^{t-6})$

Substation the equations (1.42) −(1.50) in to system (1.40), we obtain

$-\frac{5}{h}p_{1,\ell-2}-\frac{55}{h}p_{1,\ell-1}+\frac{55}{h}p_{1,\ell+1}+\frac{5}{h}p_{1,\ell+2}+t^2(=p_{1,\ell-2}+26p_{1,\ell-1}+$
$66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2})=(t^2+1)e^t-e^{(t+1)}+\frac{1}{3}(e^t-e^{t-3})\Big(\big(p_{2,\ell-2}+$
$26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2}\big)^2\Big)+\frac{1}{6}(e^t-e^{t-6})\Big(\big(p_{3,\ell-2}+$
$26p_{3,\ell-1}+66p_{3,\ell}+26p_{3,\ell+1}+p_{3,\ell+2}\big)^2\Big)$

(1.51)

$p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2}+t\Big(-\frac{5}{h}p_{2,\ell-2}-\frac{55}{h}p_{2,\ell-1}+$
$\frac{55}{h}p_{2,\ell+1}+\frac{5}{h}p_{2,\ell+2}\Big)=(t+2)e^{2t}-2e^t+\frac{1}{3}(e^t-e^{t-3})\Big(\big(p_{2,\ell-2}+$
$26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2}\big)^2\Big)+\frac{1}{2}(e^t-e^{t-2})\Big(\big(p_{1,\ell-2}+$
$26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2}\big)^2\Big)$

(1.52)

$p_{3,\ell-2}+26p_{3,\ell-1}+66p_{3,\ell}+26p_{3,\ell+1}+p_{3,\ell+2}+2t\Big(-\frac{5}{h}p_{3,\ell-2}-$
$\frac{55}{h}p_{3,\ell-1}+\frac{55}{h}p_{3,\ell+1}+\frac{5}{h}p_{3,\ell+2}\Big)=(2t+3)e^{3t}-2e^t+\frac{1}{2}(e^t-$
$e^{t-2})\Big(\big(p_{1,\ell-2}+26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2}\big)^2\Big)+\frac{1}{4}(e^t-$
$e^{t-4})\Big(\big(p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2}\big)^2\Big)$

(1.53)

We rewrite the equations (1.51),(1.52), and (1.53) as the system of matrix, and use LU matrix factorization to solve this system to evaluate the value of the coefficients $p_{i\ell-2}, p_{i\ell-1}, p_{i\ell}, p_{i\ell+1}, p_{i\ell+2}$ at $t=t_0,t_1,\ldots,t_{10}$, where $i=1,2,3$

and the approximate solution $u(t), v(t)$ and $w(t)$ can be found from

$u(t)=\sum_{e=-3}^{+3}p_{1,e}B\mathbb{S}_e^5(t)$, $v(t)=\sum_{e=-3}^{+3}p_{2,e}B\mathbb{S}_e^5(t)$,

And $w(t)=\sum_{e=-3}^{+3}p_{3,e}B\mathbb{S}_e^5(t)$

Table (1.3) displays a comparison between the theoretical and numerical results obtained using QBS for $u(t)$ in example 2, based on the least square error with h=0.1.

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T	Exact	QBS	Error(L_∞)
0.000000000000000	1.000000000000000	1.000000000000000	0.000000000000000
0.100000000000000	1.105170918075648	1.104900957561066	0.000269960514582
0.200000000000000	1.221402758160170	1.222125206739247	0.000722448579077
0.300000000000000	1.349858807576003	1.350539390043863	0.000680582467860
0.400000000000000	1.491824697641270	1.492437774258060	0.000613076616790
0.500000000000000	1.648721270700128	1.648381658786630	0.000339611913498
0.600000000000000	1.822118800390509	1.820439905729760	0.001678894660749
0.700000000000000	2.013752707470477	2.017347962194445	0.003595254723968
0.800000000000000	2.225540928492468	2.223355111546455	0.002185816946012
0.900000000000000	2.459603111156950	2.461017574917845	0.001414463760895
1.000000000000000	2.718281828459046	2.717811462129567	0.000470366329478
L.S.E.	2.429349406261739e-005		

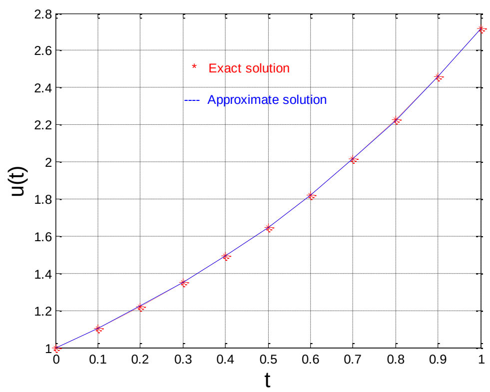
Table (1.4) displays a comparison between the theoretical and numerical results obtained using QBS for $v(t)$ in example 2, based on the least square error with h=0.1.

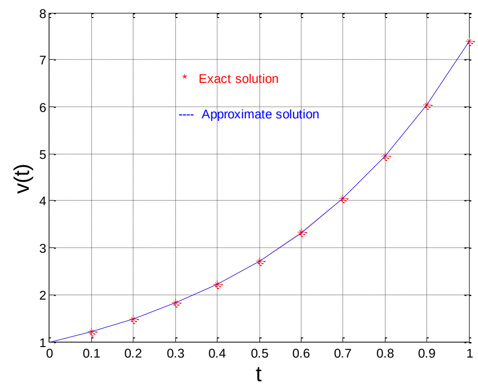
T	Exact	QSB	Error(L_∞)
0.000000000000000	1.000000000000000	1.000000000000000	0.000000000000000
0.100000000000000	1.221402758160170	1.221161823831211	0.000240934328959
0.200000000000000	1.491824697641270	1.491745850755535	0.000078846885736
0.300000000000000	1.822118800390509	1.821572806401336	0.000545993989173
0.400000000000000	2.225540928492468	2.225052017055391	0.000488911437077
0.500000000000000	2.718281828459046	2.718816030039497	0.000534201580451
0.600000000000000	3.320116922736547	3.319632060587300	0.000484862149247
0.700000000000000	4.055199966844675	4.054557695600595	0.000642271244080
0.800000000000000	4.946032424395115	4.945279966426286	0.000752457968829
0.900000000000000	6.049647464412947	6.048318518614160	0.001328945798787
1.000000000000000	7.389056098930650	7.387994461123630	0.001061637807013
L.S.E.	4.993749759327024e-006		

Table (1.5) displays a comparison between the theoretical and numerical results obtained using QBS for $w(t)$ in example 2, based on the least square error with h=0.1.

T	Exact	QSB	Error(L_∞)
0.000000000000000	1.000000000000000	1.000000000000000	0.000000000000000
0.100000000000000	1.349858807576003	1.349189511673419	0.000669295902584
0.200000000000000	1.822118800390509	1.821181422992235	0.000937377398274
0.300000000000000	2.459603111156949	2.458607905325421	0.000995205831528
0.400000000000000	3.320116922736548	3.320381755679806	0.000264832943257
0.500000000000000	4.481689070338065	4.481350878900778	0.000338191437287

0.600000000000000	6.049647464412945	6.049296683322290	0.000350781090655
0.700000000000000	8.166169912567646	8.165421823233722	0.000748089333925
0.800000000000000	11.023176380641605	11.023457084168742	0.000280703527138
0.900000000000000	14.879731724872837	14.876492940363732	0.003238784509104
1.000000000000000	20.085536923187668	20.094819415426169	0.009282492238501
L.S.E.	9.991744472469092e-005		





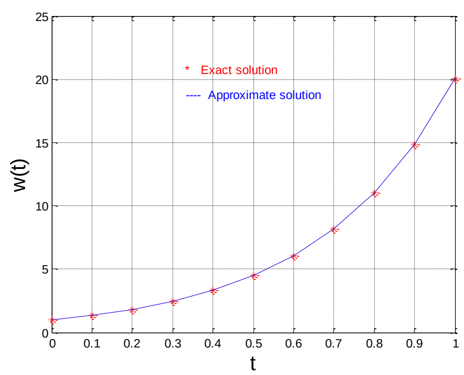


Fig.(1.2) displays a comparison between the exact and numerical solutions obtained using QBS for $u(t), v(t)$, and $w(t)$ in example 2, based on the least square error with h=0.1

Example (3):

Consider the following system of multi−type two nonlinear Fredholm integro−differential equations of the second kind.

$$\left.\begin{aligned} u''(t) + u'(t) + tu(t) &= tcos(t) + (t-6)sin(t) + \int_0^{\pi} cos(t-y)[u^2(y) + v^2(y)]dy \\ v''(t) + v'(t) + 3v(t) &= 5cos(t) - 3sin(t) + \int_0^{\pi} sin(t-y)[u^2(y) + v^2(y)]dy \end{aligned}\right\} \quad (1.54)$$

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where $$u(0)=1, u'(0)=1, v(0)=1, v'(0)=-1 \quad (1.55)$$

The exact solution are $u(t)=cos(t)+sin(t)$ and $v(t)=cos(t)-sin(t)$

$$u(t)=QB_1(t)=\mathcal{p}_{1,\ell-2}+26\mathcal{p}_{1,\ell-1}+66\mathcal{p}_{1,\ell}+26\mathcal{p}_{1,\ell+1}+\mathcal{p}_{1,\ell+2} \quad (1.56)$$

$$\frac{d}{dt}u(t)=\frac{d}{dt}QB_1(t)=-\frac{5}{h}\mathcal{p}_{1,\ell-2}-\frac{55}{h}\mathcal{p}_{1,\ell-1}+\frac{55}{h}\mathcal{p}_{1,\ell+1}+\frac{5}{h}\mathcal{p}_{1,\ell+2} \quad (1.57)$$

$$\frac{d^2}{dt^2}u(t)=\frac{d^2}{dt^2}QB_1(t)=\frac{20}{h^2}\mathcal{p}_{1\ell-2}+\frac{40}{h^2}\mathcal{p}_{1\ell-1}-\frac{120}{h^2}\mathcal{p}_{1\ell}+\frac{40}{h^2}\mathcal{p}_{1\ell+1}+\frac{20}{h^2}\mathcal{p}_{1\ell+2} \quad (1.58)$$

$$v(t)=QB_2(t)=\mathcal{p}_{2,\ell-2}+26\mathcal{p}_{2,\ell-1}+66\mathcal{p}_{2,\ell}+26\mathcal{p}_{2,\ell+1}+\mathcal{p}_{2,\ell+2} \quad (1.59)$$

$$\frac{d}{dt}v(t)=\frac{d}{dt}QB_2(t)=-\frac{5}{h}\mathcal{p}_{2,\ell-2}-\frac{55}{h}\mathcal{p}_{2,\ell-1}+\frac{55}{h}\mathcal{p}_{2,\ell+1}+\frac{5}{h}\mathcal{p}_{2,\ell+2} \quad (1.60)$$

$$\frac{d^2}{dt^2}v(t)=\frac{d^2}{dt^2}QB_2(t)=\frac{20}{h^2}\mathcal{p}_{2\ell-2}+\frac{40}{h^2}\mathcal{p}_{2\ell-1}-\frac{120}{h^2}\mathcal{p}_{2\ell}+\frac{40}{h^2}\mathcal{p}_{2\ell+1}+\frac{20}{h^2}\mathcal{p}_{2\ell+2} \quad (1.61)$$

$$u^2(t)=(QB_1(t))^2=\left(\mathcal{p}_{1,\ell-2}+26\mathcal{p}_{1,\ell-1}+66\mathcal{p}_{1,\ell}+26\mathcal{p}_{1,\ell+1}+\mathcal{p}_{1,\ell+2}\right)^2 \quad (1.62)$$

$$v^2(t)=(QB_2(t))^2=\left(\mathcal{p}_{2,\ell-2}+26\mathcal{p}_{2,\ell-1}+66\mathcal{p}_{2,\ell}+26\mathcal{p}_{2,\ell+1}+\mathcal{p}_{2,\ell+2}\right)^2 \quad (1.63)$$

And $\int_0^\pi \cos(t-y)\,dy=[-\sin(t-y)]_0^\pi=-sin(t-\pi)+sin(t)$

$\int_0^\pi \sin(t-y)\,dy=[\cos(t-y)]_0^\pi=cos(t-\pi)-cos(t)$

Substation the equations (1.56)–(1.63)in to system (1.54), we obtain

$$\frac{20}{h^2}\mathcal{p}_{1\ell-2}+\frac{40}{h^2}\mathcal{p}_{1\ell-1}-\frac{120}{h^2}\mathcal{p}_{1\ell}+\frac{40}{h^2}\mathcal{p}_{1\ell+1}+\frac{20}{h^2}\mathcal{p}_{1\ell+2}-\frac{5}{h}\mathcal{p}_{1,\ell-2}-\frac{55}{h}\mathcal{p}_{1,\ell-1}+\frac{55}{h}\mathcal{p}_{1,\ell+1}+\frac{5}{h}\mathcal{p}_{1,\ell+2}+t\left(\mathcal{p}_{1,\ell-2}+26\mathcal{p}_{1,\ell-1}+66\mathcal{p}_{1,\ell}+26\mathcal{p}_{1,\ell+1}+\mathcal{p}_{1,\ell+2}\right)=tcos(t)+(t-6)sin(t)+(-sin(t-\pi)+sin(t))\left(\left(\mathcal{p}_{1,\ell-2}+26\mathcal{p}_{1,\ell-1}+66\mathcal{p}_{1,\ell}+26\mathcal{p}_{1,\ell+1}+\mathcal{p}_{1,\ell+2}\right)^2+\left(\mathcal{p}_{2,\ell-2}+26\mathcal{p}_{2,\ell-1}+66\mathcal{p}_{2,\ell}+26\mathcal{p}_{2,\ell+1}+\mathcal{p}_{2,\ell+2}\right)^2\right)$$

(1.64)

$$\frac{20}{h^2}p_{2\ell-2}+\frac{40}{h^2}p_{2\ell-1}-\frac{120}{h^2}p_{2\ell}+\frac{40}{h^2}p_{2\ell+1}+\frac{20}{h^2}p_{2\ell+2})-\frac{5}{h}p_{2,\ell-2}-\frac{55}{h}p_{2,\ell-1}+\frac{55}{h}p_{2,\ell+1}+\frac{5}{h}p_{2,\ell+2}+3(p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2})=5cos(t)-3sin(t)+(cos(t-\pi)-cos(t))\left(\left(p_{1,\ell-2}+26p_{1,\ell-1}+66p_{1,\ell}+26p_{1,\ell+1}+p_{1,\ell+2}\right)^2+\left(p_{2,\ell-2}+26p_{2,\ell-1}+66p_{2,\ell}+26p_{2,\ell+1}+p_{2,\ell+2}\right)^2\right) \quad (1.65)$$

We rewrite the equations (1.64) and (1.65) as the system of matrix, and use LU matrix factorization to solve this system to evaluate the value of the coefficients $p_{i\ell-2}, p_{i\ell-1}, p_{i\ell}, p_{i\ell+1}, p_{i\ell+2}$ at $t=t_0,t_1,\ldots,t_{10}$, where $i=1,2$

and the approximate solution $u(t)$ and $v(t)$ can be found from

$$u(t)=\sum_{e=-3}^{+3}p_{1,e}B\mathbb{S}_e^5(t)$$

and $v(t)=\sum_{e=-3}^{+3}p_{2,e}B\mathbb{S}_e^5(t)$,

Table (1.6) displays a comparison between the theoretical and numerical results obtained using QBS for $u(t)$ in example 3, based on the least square error with h=0.1.

T	Exact	QBS	Error(L_∞)
0.000000000000000	1.000000000000000	1.000000000000000	0.000000000000000
0.100000000000000	1.094837581924854	1.095225180862248	0.000387598937394
0.200000000000000	1.178735908636303	1.178236451287586	0.000499457348717
0.300000000000000	1.250856695786946	1.250585492328722	0.000271203458224
0.400000000000000	1.310479336311536	1.310378573446287	0.000100762865249
0.500000000000000	1.357008100494576	1.356978305314501	0.000029795180075
0.600000000000000	1.389978088304714	1.390126019033501	0.000147930728787
0.700000000000000	1.409059874522180	1.409314225311100	0.000254350788920
0.800000000000000	1.414062800246688	1.413961269551550	0.000101530695138
0.900000000000000	1.404936877898148	1.404893674699979	0.000043203198169
1.000000000000000	1.381773290676036	1.382470898082720	0.000697607406683
L.S.E.	1.069691719567815e-006		

Table (1.7) displays a comparison between the theoretical and numerical results obtained using QBS for $v(t)$ in example 3, based on the least square error with h=0.1.

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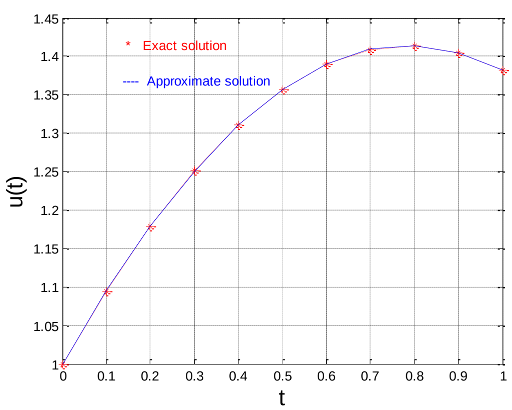
T	Exact	QBS	Error(L_∞)
0.000000000000000	1.000000000000000	1.000000000000000	0.000000000000000
0.100000000000000	0.895170748631198	0.895198065056720	0.000027316425522
0.200000000000000	0.781397247046180	0.781456916778587	0.000059669732406
0.300000000000000	0.659816282464266	0.659866532975597	0.000050250511331
0.400000000000000	0.531642651694235	0.531681429161264	0.000038777467030
0.500000000000000	0.398157023286170	0.398146653018486	0.000010370267683
0.600000000000000	0.260693141514643	0.259878893460428	0.000814248054215
0.700000000000000	0.120624500046797	0.120751445614610	0.000126945567813
0.800000000000000	-0.020649381552357	-0.020555383172683	0.000093998379674
0.900000000000000	-0.161716941356819	-0.161422266230619	0.000294675126200
1.000000000000000	-0.301168678939757	-0.300406575408598	0.000762103531159
L.S.E.	1.364029000926428e-006		



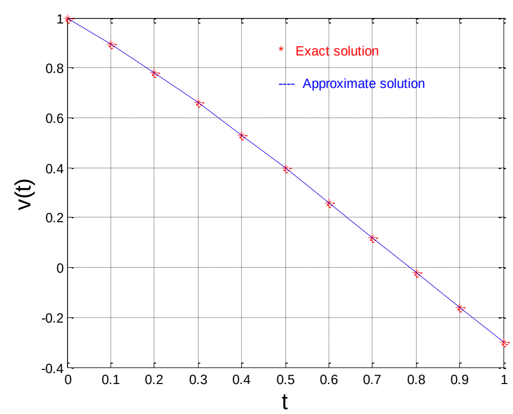


Fig.(1.3) displays a comparison between the exact and numerical solutions obtained using QBS for $u(t)$, and $v(t)$ in example 3, based on the least square error with h=0.1

5. Conclusions:

In this study, we successfully applied the Quintic B−spline method to solve systems of nonlinear Fredholm integro−differential equations of the second kind. The method was validated through three numerical examples, demonstrating high accuracy and efficiency. The comparison between theoretical and numerical solutions using maximum absolute error and least square error norms confirms the reliability of the Quintic B−spline approach. The method's ability to handle complex integro−differential equations makes it a valuable tool for various applications in scientific and engineering fields. The results highlight the potential of Quintic B-splines in providing precise numerical solutions where analytical methods are challenging to apply.

6. Recommendations:

1. Further Research: Future studies should explore the application of the Quintic B-spline method to other types of integro-differential equations and investigate its performance with different boundary and initial conditions.
2. Software Implementation: Developing software tools that incorporate the Quintic B-spline method can make this technique more accessible to researchers and engineers, facilitating its use in practical applications.
3. Comparison with Other Methods: Conducting comparative studies with other numerical methods, such as finite element and finite difference methods, can provide deeper insights into the strengths and limitations of the Quintic B-spline approach.
4. Higher-Dimensional Problems: Extending the method to solve higher-dimensional integro-differential equations can open new avenues for its application in more complex real-world problems.
5. Adaptive Algorithms: Developing adaptive algorithms that adjust the B-spline parameters based on the problem's complexity can enhance the method's efficiency and accuracy.
6. By addressing these recommendations, the potential and applicability of the Quintic B-spline method in solving complex integro-differential equations can be further realized and optimized.

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