Some Concepts and Their Relationships with Neutrosophic Quasi-Frobenius Rings

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Abstract

Within this research, we will study the relationships between Neutrosophic torsionless cyclic $(R \cup I)$-module and Neutrosophic quasi-Frobenius rings. Some implications yeld cyclic $(R \cup I)$-module and hence Neutrosophic quasi-Frobenius ring have been presented. In addition we study the relations between isomorphic Neutrosophic Noetherian ring and Neutrosophic quasi-Frobenius (Q-F) ring. Additionally, we present an important relation between Neutrosophic torsionless cyclic module, projective module and Neutrosophic injective module which through we get an Q-F ring. Finally, we study some concepts such as; Neutrosophic hollow module, Neutrosophic local module and Neutrosophic simple module which its relationship to the quasi-Frobenius rings.

Keywords: Quasi-Frobenius rings, Noetherian rings, torsionless, cyclic module, Hollo Module.

المستخلص

في هذا البحث، ندرس العلاقات بين المقاسات النيوتروسوفية الدورية الخالية من الالتواء (Neutrosophic torsionless cyclic module) والحلقات النيوتروسوفية شبه فروبينيوس (Neutrosophic quasi-Frobenius rings). وقد تم تقديم بعض النتائج التي تؤدي إلى تكوّن مقاسات دورية، ومن ثم إلى حلقة نيوتروسوفية شبه فروبينيوس. بالإضافة إلى ذلك، ندرس العلاقات

بين الحلقة النيوتروسوفية النويثيرية المتماثلة (isomorphic Neutrosophic Noetherian ring)، والحلقة النيوتروسوفية شبه فروبينيوس (Q-F ring). كما نعرض علاقة مهمّة بين المقاسات النيوتروسوفية الدورية الخالية من الالتواء، والمقاسات النيوتروسوفية الإسقاطية (Neutrosophic projective module)، والمقاسات النيوتروسوفية الحقنية (Neutrosophic injective module)، والتي من خلالها يمكن الحصول على حلقة Q-F. وأخيرًا، ندرس بعض المفاهيم مثل: المقاسات النيوتروسوفية الجوفاء (Neutrosophic hollow module)، والمقاسات النيوتروسوفية المحلية (Neutrosophic local module)، والمقاسات النيوتروسوفية البسيطة (Neutrosophic simple module)، وعلاقتها بحلقات شبه فروبينيوس.

1. Introduction

A Neutrosophic Artinian ring is a Neutrosophic ring that satisfies the descending chain condition on (one-sided) ideals; that is, there is no infinite descending sequence of ideals. A Neutrosophic ring $R \cup I$ is referred to as quasi-Frobenius if it is (left or right) self-injective and (left or right) Artinian, equivalently, on which side it is self-injective and which side it is Noetherian. A right n-injective ring R is defined as one in which each homomorphism of an n-generated right ideal to R extends into an endomorphism of R. This definition is one of several generalizations of the notion of self-injective rings. [1]. The concept of rings that are Frobenius and quasi-Frobenius are generalized by many authors. In (1941), Nakayama, Tadasi presented a study on Frobeniusian algebra [2]. In (1950), Ikeda, Masatoshi and Tadasi Nakayama submitted a study about supplementary remarks on Frobeniusian algebras [3]. In (1956), the outher studied both Hiroyuki Tachikawa, Kiiti Morita, and Morita Quasi Frobenius rings, character modules, and free module submodules [4]. In (1958), Dieudonne', Jean provided some remarks about quasi-Frobenius rings [5]. In (1964), E. A. Walker and Carl Faith proposed the concept about quasi-Frobenius rings on characterizations [6]. And in (1996), Dinh van huynh demonstrated in her study A Note on quasi- Frobenius rings [7]. Sets of fuzzy and Sets of intuitionistic fuzzy have a generalization which is the Neurosophic set. According to Neutrosophic reasoning, There are three levels to a proposition: truth (T), indeterminacy (I), and falsehood (F). There is a membership function for each component level of involvement in the uncertainty issue, according to fuzzy set theory [8]. After that, in 1983, K. Atanassov expanded fuzzy sets to include intuitionistic fuzzy sets on universe X. In these sets, in addition to Membership degree $\mu_A(x_0) \in [0,1]$ for any element x_0 to set A, non-members may also get a degree to function $\nu_A(x_0) \in [0,1]$ that are present, where $x_0 \in X$, $\mu_A(x_0) + \nu_A(x_0) \leq 1$ [9].

2. Basic Concepts

In the present section, some fundamental terms that will be used later on are defined.

Definition 2.1 [10]. Consider X as an universal set. Then a fuzzy set A within X consists of ordered pairs; $A := \{(x, \mu_A(x)) : x \in X\}$ s.t, $\mu_A : X \longrightarrow [0,1]$, that is known as the function for membership with $x \in X$, the value from μ_A(x) represents the grade of membership of x in A.

Example 2.2: The universal set X is the group of people. The issue of whether person x is young is addressed to that extent by the definition of B fuzzy subset young? To each person in the universal set, We must allocate a degree for membership inside the fuzzy subset "young." The most straightforward method to do this is using the function of membership based on the individual's age.

$$\mu_B(x) = \begin{cases} 1\,, & age(x) \leq 20 \\ (30 - age(x))/10\,, & 20 \leq age(x) \leq 30 \\ 0\,, & age(x) > 30 \end{cases}$$

Definition 2.3 [11]. Assume that G is any group. Then this mapping μ: $G \rightarrow [0\ 1]$ forms the fuzzy group if $\forall\, a, b \in G$:

1) $\mu(ab) \geq min \{ \mu (a), \mu(b)\}$.
2) $\mu(a^{-1}) = \mu (a)$.

Example 2.4: Here we have the set S, which contains all possible random variables for the space of probabilities (Ω, φ, P) and suppose that A is any subset of the reals that is Borel and includes all subgroups from the reals that are addizable. Note that S is a groupoid when added pointwise. (The extra structure is not necessary for our current goals, even if S is a group.) A function $\varphi_A: S \longrightarrow [0,1]$ define by $\varphi_A(X) = P\{w \in \Omega : X(w) \in A\} = P[X^{-1}(A)]$. Therefore, φ_A represents the probabilities that X is "in" the subgroup A. A fuzzy subgroupoid is an obvious choice for the function φ_A.

Definition 2.5 [12]. Assume that R is a ring, A is a fuzzy set of R, we will refer to A as a fuzzy ring of R, if
1) $A(a - b) \geq A(a) \wedge A(b), \forall\, a, b \in R$.
2) $A(ab) \geq A(a) \wedge A(b), \forall\, a, b \in R$.

Example 2.6: Consider the set $R = \{0, x, y\}$, which contains a binary operation $(\cdot)$ and a hyperoperation $(+)$ as follows:

$+$	0	x	y
0	$\{0\}$	$\{x\}$	$\{y\}$
x	$\{x\}$	$\{0, x, y\}$	$\{x, y\}$
y	$\{y\}$	$\{x, y\}$	$\{0, x, y\}$

$\cdot$	0	x	y
0	0	0	0
x	0	x	y
y	0	x	y

Consequently a ring $(R, +, \cdot)$ is a hypernear. A fuzzy set $\mu : R \longrightarrow [0,1]$ define by $\mu(x) = \mu(y) = 0.5$ and $\mu(0) = 1$. Checking that μ is a fuzzy subhypernear ring from R may be done using basic computations.

Definition 2.7 [13]. Assume that R is a ring. We referre to the fuzzy subset λ of R fuzzy ideal in R if $\forall\, x, y \in R$,

1) $\lambda\,(x - y) \geq\, min\,\{\,\lambda(x)\,,\ \lambda(y)\}$.

2) $\lambda\,(xy) \geq\, max\,\{\,\lambda(x)\,,\ \lambda(y)\}$.

Example 2.8: Consider R is a ring. A fuzzy subset λ define by: $\lambda(x) = r, \forall\, x \in R$ and $r \in [0,1]$, then λ is fuzzy ideal in R.

Note: If we replace $[0, 1]$ with $\{0, 1\}$ in the above definition, then a fuzzy ideal is just the usual real ideal.

Definition 2.9 [14]. Assume that a fixed set A is non-empty. The Neutrosophic set S Object having the format: $S = \{\langle A, \mu_S(a), \sigma_S(a), \gamma_S(a)\rangle : a \in A\}$ where $\mu_S(a), \sigma_S(a)$ and $\gamma_S(a)$ are represent a membership function's degree namely $\mu_S(a)$, $\sigma_S(a)$ represents the degree of indeterminacy while γ_S(a) represents an degree from non-membership for any $a \in A$ of S.

Example 2.10: Assuming the universe from discourse $U = \{x_1, x_2, x_3\}$, where x_1 describes the capacity, x_2 describes the reliability and x_3 represents the item prices Another such assumption is that that values of x_1, x_2 and x_3 located in [0,1] These are derived from surveys completed by professionals. Professionals have the option to enforce their views in three areas: the degree to which things' qualities may be explained by their goodness, indeterminacy, or poverty. Assume that A is a set of Neutrosophic (NS) on U, where, $A = \{< x_1\,, (0.3, 0.5\,, 0.6) >\,, < x_2\,, (0.3, 0.2\,, 0.3) >, < x_3\,, (0.3, 0.5\,, 0.6) >\}$, such that 0.3 is the degree for capacity goodness, degree for capability indeterminacy is 0.5 and 0.6 is the degree of capacity falsehood, etc.

Definition 2.11 [15]. Assume any ring as R. Another ring that may be generated underneath the operations of R by R and I is the Neutrosophic ring $(R \cup I)$.

Example 2.12: Assume that Z be any ring of integers; $(Z \cup I) = \{z_1 + z_2 I :$ $z_1, z_2 \in Z\}$. An integer ring referred as the Neutrosophic ring is $(Z \cup I)$. Also $Z \subseteq (Z \cup I)$.

Definition 2.13 [16]. Let $\mu = (\mu^L, \mu^J, \mu^E)$ be any non-empty Neutrosophic subset of a β-semiring A (i.e. anyone of $\mu^L(a), \mu^J(a)\ or\ \mu^E(a) \neq 0;\ a \in A$). Then μ referred to as a Neutrosophic left ideal of A, $\forall\ a, b \in A\ and\ \psi \in \beta$ if:

i. $\mu^L(a + b) \geq min\{\mu^L(a), \mu^L(b)\}, \mu^L(a\,\psi\,b) \geq \mu^L(b)$

ii. $\mu^J(a + b) \geq \frac{\mu^J(a) + \mu^J(b)}{2}, \mu^J(a\,\psi\,b) \geq \mu^J(b)$

iii. $\mu^E(a + b) \leq max\{\mu^E(a), \mu^E(b)\}, \mu^E(a\,\psi\,b) \leq \mu^E(b)$.

In the same way, the Neutrosophic right ideal of A may be defined.

Example 2.14: A and γ are the semigroups of all not positive integers and all not positive even integers respectively that are additive and commutative. After that A is a β -semiring if $a\ \psi\ \ b$ represents the represents multiplication from integers a_1, ψ, a_2 where $a_1, a_2 \in A$ and $\psi \in \beta$. Define a Neutrosophic subset μ of A as follows

$$\mu(w) = \begin{cases} (1, 0, 0)\ if\ w = 0 \\ (0.8, 0.3, 0.4)\ if\ w\ is\ even \\ (0.3, .02, 0.7)\ if\ w\ is\ odd \end{cases}$$

Then μ of A is an ideal that is Neutrosophic.

Definition 2.15 [17]. Remember that in order for a ring A to be considered Noetherian, it must meet the following three comparable requirements:

(1) maximum elements (the maximum condition) exist in all nonempty sets from ideals of A.

(2) Every ascending sequences of ideals are stationary (the ascending chain condition (A.C.C.))

(3) Every ideal of A is f-generated.

Definition 2.16 [18]. An R-module is considered free if it is isomorphic to another R-module of the type $\bigoplus_{i\in I} M_i$, where each $M_i \cong R$ (in the sense that it is an R-module). The notation for such a module is $R^{(I)}$.

Definition 2.17 [6]. Let R be a ring, M be R-module and defined $\sigma_M : M \to M^{**}$ as: $[\sigma_M(m_0)](f) = f(m_0)$, $f \in M^*$, $m_0 \in M$. Then M is torsionless iff σ_M is a R-monomorphism.

Definition 2.18 [19]. Assuming that M be an R-module also let $m_i \in M, \forall\, i \in I$, s.t. I represents some indexing set. Now if $M = \sum_{i\in I} Am_i$, then $\{m_i | i \in I\}$ is referred to as the set generators of M.

Definition 2.19 [19]. For R-modules M with an finite number from generators, so we say to be it is finitely generated.

Definition 2.20 [20]. If $Z(M) = M$, such that $Z(M) = \{\, m \in M : mI = (0)$, for some essential ideal I of $R\}$. Then R-module M is said to be singular.

Definition 2.21 [21]. If for every homomorphisms R-module $\phi: N \longrightarrow W$ and $\psi: N \longrightarrow M$ where ϕ is injective, there exists an R-linear homomorphism $\Omega: W \longrightarrow M$ such that $\Omega \circ \phi = \psi$. Then R-module M is said to be injective.

Definition 2.22 [22]. Let $M \oplus K$ is the free R-module s.t. K is each module on R. After that an R-module M is said to be projective.

Definition 2.23 [23]. All proper submodules of the non-zero R-module M must be small submodules of M for M to be a hollow module.

Definition 2.24 [24]. The R-module M that is non-trivial is referred to as semi hollow if all proper submodule from M is a semi small submodule from M.

Definition 2.25 [18]. An R-module M is referred to as local lifting if a module M have the maximal submodule N that is unique. There are a submodules A and B of N where $M = A \oplus B$ and $N \cap B$ is small submodule of B.

3. Auxiliary results

We begin with the following lemmas which needs its in the our main results.

Lemma 3.1. [25]. Consider a ring R is Noetherian. Hence, the following criteria are equivalent:

1) R is an Q-F ring.
2) All R-module is submodule from free R-module.
3) All R-module is torsionless.
4) All f-generated R-module is torsionless.
5) All f-generated R-module represents submodule of an free R-module.

Lemma 3.2. [26]. Assume that C is an cyclic R-module. We say $\mathrm{C} = R/\ A$ such that A is an ideal in R.

1) $\mathrm{C}^* \cong ann\ (A)$ as an R-modules.
2) C is torsionless if and only if A is annihilator.
3) C is reflexive if and only if A is annihilator and all R-homomorphism $ann(A) \longrightarrow R$ yields the result of multiplying by an element of R.

Lemma 3.3. [27]. The ring R is said to be Q-F iff it is a ring that is Noetherian in addition to relationships $s(d(I)) = I$ and $d(s(J)) = J$ are applicable for all ideals J and I from R.

Lemma 3.4. [28]. Assume that M is a module. Then a module M is considered an f-generated hollow module iff it is has a unique maximum submodule and is cyclic.

Lemma 3.5. [29]. The Noetherian state of $R/socle(R)$ and the injectivity from every simple $R/socle(R)$-modules follow based on the reality that every cyclic singular R-modules are injective.

Lemma 3.6. [30].

Consider the ring R where each cyclic is injective. In such case, R is an Artinian semi simple.

Lemma 3.7. [31]. These two options are analogous:

1) All cyclic modules in a ring R are extensions from injective modules in a projective module.
2) It is true that all singular module certainly injective.

Lemma 3.8. [18]. Assume that the module M is an f-generated. Then M is local lifting iff M has a unique maximum submodule and is cyclic.

Lemma 3.9. [18]. Assume that M is an R-module. M/N is local lifting module if M is local lifting module for all N proper submodule of M.

Lemma 3.10. [18]. Every a local lifting module is indecomposable module.

Lemma 3.11. [18]. Assume that M is R-module. A module M is local lifting iff M is a lifting and cyclic module.

4. Main Results

In this part, we present some different rings and modules and their relationship with the Neutrosophic Quasi-Frobenius Ring.

Definition 4.1. A Neutrosophic ring $(R \cup I)$ is referred to as Neutrosophic self-injective ring if $(R \cup I)$ represents injective as the Neutrosophic module on itself (i.e., $(R \cup I)$ is Neutrosophic injective as a left or right $(R \cup I)$-module).

Example 4.2: Let $(R \cup I) = (Z \cup I)/n(Z \cup I)$ for any integer $n \geq 1$. Then $(R \cup I)$ is a Neutrosophic self-injective. Because $(Z \cup I)/n(Z \cup I)$ is a finite Neutrosophic ring, hence Neutrosophic Artinian, and it is Neutrosophic quasi-Frobenius ring implies that it is Neutrosophic self-injective. In addition, as a Neutrosophic module over itself, every homomorphism from as Neutrosophic ideal into $(R \cup I)$ extends to all of $(R \cup I)$. Thus, $(Z \cup I)/n(Z \cup I)$ is a Neutrosophic self-injective ring.

Definition 4.3. A Neutrosophic ring $(R \cup I)$ is said to be Neutrosophic Artinian if $(R \cup I)$ satisfies the descending chain condition (D.C.C): all descending chain from Neutrosophic ideals of $(R \cup I)$, $(K_0 \cup I) \supseteq (K_1 \cup I) \supseteq \cdots \supseteq (K_n \cup I) \supseteq (K_{n+1} \cup I) \supseteq \cdots$ is stationary.

Example 4.4: Assume $(K(t) \cup I)$ is the Neutrosophic polynomial ring in the variable t accompanied with coefficients in a Neutrosophic field K. Then for each positive integer n, the residue ring consisting of $(K(t) \cup I)/(t^n)$ is both Neutrosophic Artinian and Noetherian. Reason being, a vector space $(K(t) \cup I)/(t^n)$ is finite and has n dimensions.

Definition 4.5. Any Neutrosophic ring that does not have an infinite escalating chain from right (or left) Neutrosophic ideals is referred to as left (or right) Neutrosophic Noetherian.

In this particular instance from above definition we say that the ring satisfy the (A.C.C) on the left (or right) Neutrosophic ideals.

Note: A Neutrosophic ring $(R \cup I)$ is called Neutrosophic Noetherian if it is Neutrosophic Noetherian both left and right.

Example 4.6: Consider the Neutrosophic ring from integers $(\mathrm{Z} \cup I)$. Look at an Neutrosophic ideal in $(\mathrm{Z} \cup I)$ of the form $(K \cup I) = (6I) = \{\, 6nI \mid nI \in (Z \cup I)\}$ are f-generated by the single element $6I$. In fact, every Neutrosophic ideal in $(Z \cup I)$ is generated by one integer. Therefore, all Neutrosophic ideals are f-generated. Hence $(\mathrm{Z} \cup I)$ is an Neutrosophic Noetherian ring.

Definition 4.7. If $(R \cup I)$ is a Neutrosophic self-injective and Neutrosophic Artinian ring, or if it is Neutrosophic self-injective and Neutrosophic Noetherian ring, then it can be called Neutrosophic Q-F ring.

Example 4.8: Where m is any positive integer and $(R \cup I)$ represents an Neutrosophic integers ring. After that, an Neutrosophic quotient integers ring modulo m is Neutrosophic Q-F ring because $(R \cup I)$ is an Neutrosophic commutative, finite and primary ideal.

Theorem 4.9. An Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Q-F iff all cyclic $(R \cup I)$-module is torsionless.

Proof: Assuming that $(R \cup I)$ is Neutrosophic Q-F, we know from [32, Exercise 15.7] also Lemma 3.1, all Neutrosophic $(R \cup I)$-module is torsionless.

Conversely, assuming all Neutrosophic cyclic $(R \cup I)$-module is torsionless. Through Lemma 3.2, we have each Neutrosophic ideal from $(R \cup I)$ is annihilator. Considering $(R \cup I)$ is Neutrosophic Noetherian, $(R \cup I)$ is Neutrosophic Q-F from Lemma 3.3.

Theorem 4.10. All torsionless cyclic Neutrosophic $(R \cup I)$-modules that not isomorphic into R and are injective are defined for a Neutrosophic ring $(R \cup I)$. After that $(R \cup I)$ is Q-F ring.

Proof: Assuming $(R \cup I)$ is the Neutrosophic ring also assume that $(M \cup I)$ is Neutrosophic torsionless cyclic module and it is not isomorphic into $(R \cup I)$ is injective. Then Lemma 3.5, refer to $(R \cup I)/\text{socle}(R \cup I)$ is a Neutrosophic Noetherian. If $socle(R \cup I) \neq 0$, after that $(R \cup I)/\text{socle}(R \cup I)$ is a Neutrosophic semi simple Artinian by Lemma 3.6, because all quotient of that is annihilated through $\text{socle}(R \cup I)$, and $(R \cup I)$ is not. Assume $(R \cup I) \neq \text{socle}(R \cup I)$. Let $yI \in (R \cup I)$, $yI(R \cup I)\ /socle(\ yI(R \cup I)\ )$ simple. If $socle(\ yI(R \cup I)\ )$ is length that is not finite, afterward

$socle(\ yI(R \cup I)\ ) = S \oplus T \oplus U$,

where the lengths of S, T and U are infinite. Because for $yI(R \cup I)$ then S is not direct summand, $yI(R \cup I)\ /S$ is not projective. Hence $yI(R \cup I)\ /S$ is injective. Then $T \oplus U$ embeds in $yI(R \cup I)\ /S$, and $yI(R \cup I)\ JS = E(\ T) \oplus E(\ U) \oplus K$ Regarding a few injective hulls related to U and T. Then $yI(R \cup I)\ /(S \oplus T \oplus U) \approx E(\ T)/T \oplus E(\ U)/U \oplus K$ It's not simple, an incongruity. Consequently the socle from $yI(R \cup I)$ has the length a finite. Since any simple submodule from $(R \cup I)$ is injective, the socle from $yI(R \cup I)$ is direct summand from $yI(R \cup I)$ also $yI \subset socle(R \cup I)$. Consequently $(R \cup I) = socle(R \cup I)$ is semi simple Artinian. Hence $(R \cup I)$ is Neutrosophic Noetherian ring. Consequently, through Theorem 4.9, $(R \cup I)$ is Q-F ring.

Theorem 4.11. Consider $(R \cup I)$ is the Neutrosophic ring and all cyclic module that is Neutrosophic and torsionless is the direct sum from two modules, one projective and one injective, in that ring. Then the ring $(R \cup I)$ is Q-F.

Proof: By Lemma 3.7, every Neutrosophic $(R \cup I)$-module singular is a Neutrosophic injective, and through Lemma 3.5, we have that $(R \cup I)/socle\ (R \cup I)$ is a Neutrosophic Noetherian. By the outcome from the chatter [33, Theorem 3.11], all that has to be shown is that each cyclic module that is direct sum from two modules: one projective and one Neutrosophic Noetherian. This is going to ensue if all Neutrosophic cyclic injective module is Neutrosophic Noetherian. Assume $(R \cup I)$ is a Neutrosophic injective cyclic $(R \cup I)$-module, also assume $S = socle(R \cup I)$. After that $x(R \cup I)/xS$ is cyclic $(R \cup I)/S$-module consequently Noetherian. Should xS does not have a finite length, it will decomposition into a direct sum $\bigoplus_{i=0}^{\infty} X_i$, where the length of any X_i is infinite. Let E_i be an injective hull from X_i in x$(R \cup I)$. As a result, E_i is not semi-simple and has an indefinite length as it is cyclic. Then $x(R \cup I)/xS$ includes the infinite direct sum $\bigoplus_{i=0}^{\infty} E_i/X_i$, contradicting the property that $x(R \cup I)/xS$ is Noetherian. Consequently, through Theorem 4.9, $(R \cup I)$ is Q-F ring.

Theorem 4.12. A Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Neutrosophic Q-F iff all torsionless $(R \cup I)$-module is f-generated hollow.

Proof: Assuming that $(R \cup I)$ is Neutrosophic Noetherian Q-F ring, also assume $(M \cup I)$ be a Neutrosophic torsionless $(R \cup I)$-module. After that through Theorem 4.9, we have a module $(M \cup I)$ is cyclic. We claim, $(M \cup I)$ having a unique maximal Neutrosophic submodule, say $(E \cup I)$, then $(M \cup I)$ be a Neutrosophic f-generated. Let $(L \cup I)$ be a proper Neutrosophic submodule from $(M \cup I)$ with $(K \cup I) + (D \cup I) = (M \cup I)$, where $(D \cup I)$ is the submodule of $(M \cup I)$. Now, if $(D \cup I) \neq (M \cup I)$, then $(D \cup I)$ is an proper submodule from $(M \cup I)$, and hence $(D \cup I)$ is contained in a submodule that is maximal, since

$(M \cup I)$ f-generated. However, through our claim $(M \cup I)$ has a submodule $(E \cup I)$ that is unique maximal, consequently $(K \cup I)$ is contained in $(E \cup I)$. Therefore, $(K \cup I) + (E \cup I) = (E \cup I) = (M \cup I)$, which is a contradiction. Hence, $(D \cup I) = (M \cup I)$, thus, $(K \cup I) \ll (M \cup I)$. That is, $(M \cup I)$ is a hollow module. Consequently, through Lemma 3.4, $(M \cup I)$ is f-generated hollow module.

Conversely, assuming that $(M \cup I)$ be a Neutrosophic torsionless f-generated hollow module, then

$$(M \cup I) = (R \cup I)m_1 I + (R \cup I)m_2 I + \cdots + (R \cup I)m_n I$$

for $m_i I \in (M \cup I)$ and $i = 1,2, \dots, n$ if $(M \cup I) \neq (R \cup I)m_1 I$, then $(R \cup I)m_1 I$ is a proper submodule of $(M \cup I)$, which implies that $(R \cup I)m_1 I \ll (M \cup I)$. Hence,

$$(M \cup I) = (R \cup I)m_2 I + (R \cup I)m_3 I + \cdots + (R \cup I)m_n I$$

Just keep going back to this line of reasoning until we get $(M \cup I) = (R \cup I)m_i I$ for some i. Thus $(M \cup I)$ is a cyclic module. And from the hypothesis $(M \cup I)$ is torsionless, consequently, by Theorem 4.9, $(R \cup I)$ is a Neutrosophic Q-F ring.

Theorem 4.13. An Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Neutrosophic Q-F iff all torsionless $(R \cup I)$-module is f-generated semi hollow.

Proof: Assuming that $(R \cup I)$ is an Neutrosophic Noetherian Q-F ring, and assume $(M \cup I)$ is a torsionless $(R \cup I)$-module. Then by Theorem 4.9, we have $(M \cup I)$ represents Neutrosophic cyclic $(R \cup I)$-module. After that it is Neutrosophic f-generated and consequently all proper Neutrosophic submodule of $(M \cup I)$ contained in maximal Neutrosophic submodule, but by Lemma 3.4,

$(M \cup I)$ has a Neutrosophic submodule that is unique maximal. Therefore is a Neutrosophic semihollow module.

Conversely, assuming that $(M \cup I)$ is a torsionless semihollow and a f-generated, therefore it is local. Consequently, is a Neutrosophic hollow and we have f-generated. Hence is a Neutrosophic cyclic and then by Theorem 4.9, $(R \cup I)$ is Q-F ring.

Theorem 4.14. Assuming that $(M \cup I)$ is non-zero Neutrosophic module. Then an Neutrosophic Noetherian ring $(R \cup I)$ is referred to as a Neutrosophic Q-F iff every Neutrosophic torsionless $(R \cup I)$-module is hollow and $Rad\ (M \cup I) \neq (M \cup I)$.

Proof: Assuming that a ring $(R \cup I)$ is the Neutrosophic Noetherian Q-F, and assume $(M \cup I)$ be a torsionless Neutrosophic hollow module, then by Theorem 4.9, $(M \cup I)$ represents Neutrosophic cyclic module. After that $(M \cup I)$ is the Neutrosophic f-generated module. Consequently, $(M \cup I)$ has a submodule that is maximal, which suggests that $Rad\ (M \cup I) \neq (M \cup I)$.

Conversely, assume that $(M \cup I)$ is a torsionless hollow module and Rad $(M \cup I) \neq (M \cup I)$, then Rad $(M \cup I) \ll (M \cup I)$. also through Lemma 3.4, $Rad\ (M \cup I)$ is the maximal submodule that is unique from $(M \cup I)$ and thus $(M \cup I)/Rad\ (M \cup I)$ represents simple module and hence cyclic. We claim that $(M \cup I) = (R \cup I)mI$. Let $wI \in (M \cup I)$ then $wI + Rad\ (M \cup I) \in (M \cup I)/Rad\ (M \cup I)$, and therefore there is $rI \in (R \cup I)$ s.t. $wI + Rad\ (M \cup I) = rI\big(mI + Rad\ (M \cup I)\big) = (rI)(mI) + Rad\ (M \cup I)$. i.e., $wI - (rI)(mI) \in Rad\ (M \cup I)$, which implies that $wI - (rI)(mI) = y$ for some $y \in Rad\ (M \cup I)$. Thus $wI = (rI)(mI) + y \in (R \cup I)\, mI + Rad\ (M \cup I)$, hence $(M \cup I) = (R \cup I)\, mI + Rad\ (M \cup I)$. But $Rad\ (M \cup I) \ll (M \cup I)$ implies $(M \cup I)$ cyclic $(R \cup I)$-module. Consequently, through Theorem 4.9, the ring $(R \cup I)$ is Q-F.

Theorem 4.15. Consider $(M \cup I)$ be the non-zero Neutrosophic module. Then the Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Q-F iff every Neutrosophic torsionless $(R \cup I)$-module is semihollow Plus $Rad(M \cup I) \neq (M \cup I)$.

Proof: Assuming that a ring $(R \cup I)$ is Neutrosophic Noetherian Q-F, and assuming $(M \cup I)$ is Neutrosophic torsionless semihollow module, then by Theorem 4.9, We possess a module $(M \cup I)$ is the Neutrosophic cyclic. After that $(M \cup I)$ represents f-generated $(R \cup I)$-module. Thus, $(M \cup I)$ has a submodule that is maximal, which suggests that $Rad(M \cup I) \neq (M \cup I)$.
Conversely, assume $(M \cup I)$ is an Neutrosophic torsionless semihollow and $Rad(M \cup I) \neq (M \cup I)$, therefore $(M \cup I)$ is a local Consequently $(M \cup I)$ Neutrosophic cyclic $(R \cup I)$-module. Consequently, through Theorem 4.9, $(R \cup I)$ represents Neutrosophic Q-F ring.

Theorem 4.16. A Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Q-F if all Neutrosophic torsionless $(R \cup I)$-module is simple.

Proof: Assuming that $(R \cup I)$ be the Neutrosophic Noetherian ring and assume $(M \cup I)$ is an Neutrosophic torsionless simple module and $mI \in (M \cup I)$. Both $(R \cup I)mI$ and $B = \{ cI \in (M \cup I) | \ (R \cup I)cI = 0 \}$ are submodules from $(M \cup I)$. Because $(M \cup I)$ is an Neutrosophic simple, then any of them is either 0 or $(M \cup I)$. But $(R \cup I)(M \cup I) \neq 0$ leads to B $\neq (M \cup I)$. Consequently $B = 0$, whence $(R \cup I)aI = (M \cup I)$ for every non-zero $mI \in (M \cup I)$. Therefore $(M \cup I)$ is cyclic. Consequently, through Theorem 4.9, $(R \cup I)$ represents Neutrosophic Q-F ring.

Theorem 4.17. A Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Neutrosophic Q-F iff every torsionless Neutrosophic $(R \cup I)$-module is an local.

Proof: Assuming that the Neutrosophic Noetherian ring $(R \cup I)$ is Q-F, and assume $(M \cup I)$ be an Neutrosophic torsionless $(R \cup I)$-module. After that by Theorem 4.9, we possess $(M \cup I)$ is cyclic $(R \cup I)$-module also through Theorem 4.12, $(M \cup I)$ is a f-generated hollow. Hence $(M \cup I)$ has a submodule that is maximal say $(E \cup I)$. Assume the proper submodule from $(M \cup I)$ is $(K \cup I)$, if $(K \cup I)$ is not contained in $(K \cup I)$, then $(K \cup I) + (E \cup I) = (M \cup I)$, but $(M \cup I)$ is a hollow module, thus $(E \cup I) = (M \cup I)$, consequently, a contradiction arises. This suggests that each appropriate submodule of $(M \cup I)$ is found in $(E \cup I)$, i.e., $(M \cup I)$ has a maximal submodule that is unique and contains all proper submodule from $(M \cup I)$. Hence $(M \cup I)$ is a local module.

Conversely, assuming $(M \cup I)$ is Neutrosophic torsionless local $(R \cup I)$-module then it has the maximal submodule $(E \cup I)$ that is unique by definition of local module which contains every proper submodule from $(M \cup I)$. Assuming $wI \in (M \cup I)$ with $wI \notin (E \cup I)$ afterward $(R \cup I)wI$ is a submodule from $(M \cup I)$. We argue that $(R \cup I)wI = (M \cup I)$. If not $(R \cup I)wI$ is an proper submodule from $(M \cup I)$, hence $(R \cup I)wI \leq (E \cup I)$ that means $wI \in (E \cup I)$ this results in a contradiction. Thus, $(M \cup I)$ is Neutrosophic cyclic module. Consequently, by Theorem 4.9, $(R \cup I)$ is Q-F ring.

Theorem 4.18. A Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Neutrosophic Q-F if $(M \cup I)$ is an Neutrosophic torsionless and local lifting $(R \cup I)$-module and $Rad(M \cup I) \neq (M \cup I)$.

Proof: Assuming that the Neutrosophic ring $(R \cup I)$ is an Noetherian and $(M \cup I)$ is an Neutrosophic torsionless and local lifting $(R \cup I)$-module and $Rad\ (M \cup I) \neq (M \cup I)$, thus, there is a unique maximal submodule $(N \cup I)$ of $(M \cup I)$ and each submodule from $(N \cup I)$. Here exists submodules $(E \cup I)$ and $(F \cup I)$ from $(N \cup I)$ where $(M \cup I) = (E \cup I) \oplus (F \cup I)$ and $(N \cup I) \cap (F \cup I)$ is small submodule of $(F \cup I)$. After that $(M \cup I) = (M \cup I) \oplus \{0\}$, where $\{0\}$ is submodule of $(N \cup I)$, $(N \cup I) \cap (M \cup I) = (N \cup I)$ and since $(M \cup I)$ is a local lifting module. Then $(N \cup I) \cap (M \cup I) = (N \cup I)$ is small submodule from $(M \cup I)$. Thus $(M \cup I)$ is an Neutrosophic hollow module. Hence through Theorem 4.14, $(R \cup I)$ is Q-F ring.

Theorem 4.19. Let $(M \cup I)$ represents the Neutrosophic torsionless $(R \cup I)$-module. Then a Neutrosophic Notherian ring $(R \cup I)$ is referred to as Neutrosophic Q-F if $Rad(M \cup I)$ is small and maximal Neutrosophic submodule in $(M \cup I)$.

Proof: Assuming that the Neutrosophic ring $(R \cup I)$ is an Noetherian also assume $(M \cup I)$ is an Neutrosophic torsionless $(R \cup I)$-module. Suppose that $Rad(M \cup I)$ is a small and maximal submodule in $(M \cup I)$. First we want to demonstrate that $Rad\ (M \cup I)$ is a maximal submodule in $(M \cup I)$ that is unique. suppose $(D \cup I)$ is another submodule in $(M \cup I)$ that is maximal, then $(M \cup I) = (D \cup I) + Rad\ (M \cup I)$, but $Rad\ (M \cup I) \ll (M \cup I)$ which implies that $(D \cup I) = (M \cup I)$, which is a contradiction. Thus $Rad\ (M \cup I)$ is a maximal submodule in $(M \cup I)$ that is unique. We assert ownership of all proper submodule from $(M \cup I)$ found in in $Rad\ (M \cup I)$. Assuming $(E \cup I)$ represents proper submodule from $(M \cup I)$, so if $(E \cup I)$ is not contained in $Rad\ (M \cup I)$, then $(E \cup I) + Rad\ (M \cup I) = (M \cup I)$. But $Rad\ (M \cup I) \ll (M \cup I)$ which implies that $(E \cup I) = (M \cup I)$ then we get contradiction. Consequently a module $(M \cup I)$ is Neutrosophic local. Thus, through Theorem 4.17, $(R \cup I)$ represents Neutrosophic Q-F ring.

Theorem 4.20. A Neutrosophic Noetherian ring $(R \cup I)$ is referred to as Neutrosophic Q-F iff every non-zero torsionless factor module of $(M \cup I)$ in decomposable.

Proof: Assuming that the ring $(R \cup I)$ is Neutrosophic Noetherian Q-F, also assume $(M \cup I)$ be a non-zero Neutrosophic torsionless factor module. Then through Theorem 4.9, We possess $(M \cup I)$ is Neutrosophic cyclic $(R \cup I)$-

module. Assume $(M \cup I)/(D \cup I) \neq 0$ is a factor module from $(M \cup I)$. Given the Lemma 3.4 and Lemma 3.8, we possess $(M \cup I)$ is a module for lifting locally. So $(M \cup I)/(D \cup I)$ represents module to lifting locally through Lemma 3.9. Thus through Lemma 3.10, we obtain $(M \cup I)/(D \cup I)$ be an decomposable. Conversely, Assume $(D \cup I)$ is maximal submodule from $(M \cup I)$ and assume $(L \cup I)$ is a non-zero submodule from $(D \cup I)$. suppose that $(M \cup I) = (L \cup I) + (K \cup I)$, where $(K \cup I)$ is submodule from $(M \cup I)$ through [34, lemma 1.2.10], we acquire $(M \cup I)/(L \cup I) \cap (K \cup I) \cong (M \cup I)/(L \cup I) \oplus (M \cup I)/(K \cup I)$. But $(M \cup I)/(L \cup I) \cap (K \cup I)$ is in decomposable then by second isomorphism theorem. Ether $(M \cup I)/(E \cup I) = 0$ or $(M \cup I)/(K \cup I) = 0$. Since $(L \cup I)$ is submodule of $(D \cup I)$, and $(D \cup I)$ is submodule from $(M \cup I)$. Then $(L \cup I)$ represents proper submodule from $(M \cup I)$. Hence $(M \cup I)/(L \cup I) \neq 0$ implies that $(M \cup I)/(K \cup I) = 0$ and hence $(M \cup I) = (K \cup I)$. Therefore $(L \cup I)$ is a small submodule from $(M \cup I)$. Thus $(M \cup I)$ is a Neutrosophic local lifting module and through Lemma 3.11, $(M \cup I)$ is cyclic. Thus by Theorem 4.9, ring $(R \cup I)$ is Neutrosophic Q-F.

5. Conclusion

In this research article, we studied the relation between torsionless cyclic R-module and quasi-Frobenius rings. Also we studied some relationships through which we obtained the cyclic R-module thus we obtained the Q-F ring. In addition we studied an relation between isomorphic through we obtain the Noetherian ring and thus we obtained an Q-F ring. Additionally, we studied an relation between torsionless cyclic module and projective and injective module which through we garnered the quasi-Frobenius ring. Finally, we discussed some concepts such as: the singw, local and simple module and Its relationship to the quasi-Frobenius rings.

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