Neutrosophic Hybrid Weibull Inverse Weibull distribution: Mathematical Properties with simulation and Neutrosophic Real Data Application

Kamal N. Abdullah[1], Mundher A. Khaleel [2]
[1,1] Mathematics Departments, College of Computer Science and Mathematics, Tikrit University, Iraq
[1,2] Mathematics Departments, College of Computer Science and Mathematics, Tikrit University, Iraq
[1,1] kn230022pcm@st.tu.edu.iq [1,2] mun880088@tu.edu.iq

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Abstract:

This study is concerned with finding a statistical distribution that deals with a Neutrosophic random variable and Neutrosophic parameters called Neutrosophic Hybrid Weibull Inverse Weibull (NHWIW) distribution. The basic functions of proposed distribution are found, as well as many statistical properties of distribution with an estimation of the model parameters in three different techniques, with a Monte Carlo simulation to determine the estimation efficiency of NHWIW distribution, with a comparison with three measures to determine the best method for estimation. A practical application is also conducted on two types of Neutrosophic real data, the first represented by mortality data for children under five years of age, and the second is COVID-19 in Netherlands for thirty days, where the analysis efficiency of the NHWIW distribution is determined by comparing it with six other distributions using 4 information criteria and 4 statistical measures, which showed the efficiency and flexibility of NHWIW distribution.

Keywords: HWG-family, Neutrosophic data, Bias, Cramér-von Mises, and flexibility.

1. Introduction:

Probability distributions are a fundamental tool in statistical modeling used to describe and analyze various phenomena in many fields. One of the most prominent methods recently developed to expand the scope of basic distributions is the T-X method, which provides a flexible framework for creating new distributions with improved mathematical properties, making them more accurate in representing real-world data. The methods is based on forming a family of complex distributions using transformation functions, which contributes to enhancing the flexibility and ability to deal with complex properties, such as heterogeneous or asymmetrically distributed data [1]. Examples of this method include: BIIIEE-X family [2], NOGEE−G family [3], WEE-X Family [4], OLG family [5], NGOF-G Family [6], EOIW-G Family [7], GOM-G family [8], and HOE-Φ family [9]. This study based on HWG family which was has a CDF function by form [10]:

$$F_{HWG}(x,a,b,\zeta) = 1 - e^{\left(-a\left[-\mathcal{G}(x;\varepsilon).log\left(1-\mathcal{G}(x;\varepsilon)\right)\right]^{b}\right)}, \qquad x \geq 0, a, b > 0 \qquad (1)$$

And PDF function by form:

$$f_{HWG}(x,a,b,\zeta) = ab\; g(x;\varepsilon)\left[\frac{\mathcal{G}(x;\varepsilon)}{1-\mathcal{G}(x;\varepsilon)} - log\left(1-\mathcal{G}(x;\varepsilon)\right)\right] \qquad (2)$$

$$\times\left[-\mathcal{G}(x;\varepsilon).log\left(1-\mathcal{G}(x;\varepsilon)\right)\right]^{b-1} e^{\left(-a\left[-\mathcal{G}(x;\varepsilon).log\left(1-\mathcal{G}(x,\varepsilon)\right)\right]^{b}\right)}$$

Where $\mathcal{G}(x;\varepsilon)$, and $g(x;\varepsilon)$ are CDF and PDF functions for any baseline distribution and $a, b \geq 0$ are shapes parameters for HWG family.

On other hand, the Neutrosophic logic (N.L) is a recent development in the field of fuzzy and uncertain data analysis. This logic aims to address ambiguity and uncertainty in data by introducing three main dimensions: Truth (T), falsehood (F), and indeterminacy (I). this framework provides an effective way to model data that cannot be conclusively characterized using traditional methods. There are two types of N.L: the traditional method, in which the data is divided into three parts and then dealt with by using, for example, the triangular function, in which the peaks represent the T values and the troughs represent F values and what is an between represents I values, or trapezoidal function in the same manner. As for the second method, which depends on the values of intervals and contains all parts of N.L, which called the direct method, and it's the method used in forming the proposed distribution.

Focuses on finding a statistical distribution that deals with a Neutrosophic variables and parameters. The main gap is the lack of previous research that integrates Neutrosophic data with complex distributions such as a distribution based on a hybrid family of Weibull distribution and a hybrid integral limit. Existing studies often focus on traditional data or on specific distributions without considering Neutrosophic data of an uncertain or ambiguous nature. The aim of the study is to develop a " Neutrosophic Hybrid Weibull Inverse Weibull" and test its efficiency and flexibility on real Neutrosophic data using multiple estimation methods, and compare with other distributions to determine the efficiency and flexibility in dealing with Neutrosophic data.

2. Neutrosophic Hybrid Weibull Inverse Weibull (NHWIW) distribution

Let X be a random variable, then the CDF and PDF functions for Inverse Weibull distribution has a forms [11]:

$$\mathcal{G}(x;q,p) = e^{-qx^{-p}} \quad ,q,p,x > 0 \tag{3}$$

$$g(x;q,p) = qpx^{-(p+1)}e^{-qx^{-p}} \quad ,q,p,x > 0 \tag{4}$$

Where q, p are shape parameters for Inverse Weibull distribution.

To get the CDF function for Hybrid Weibull Inverse Weibull we combine equation (1) with equation (3) by form:

$$F(x) = 1 - e^{\left(-a\left[-e^{-qx^{-p}}.log\left(1-e^{-qx^{-p}}\right)\right]^{b}\right)}, \qquad x\ a,b,q,p > 0 \tag{5}$$

The PDF function for Hybrid Weibull Inverse Weibull (HWIW) we combine equation (2) with equation (3) and (4) to get it by form:

$$f(x) = ab\ qpx^{-(p+1)}e^{-qx^{-p}}\left[\frac{e^{-qx^{-p}}}{1-e^{-qx^{-p}}} - log\left(1-e^{-qx^{-p}}\right)\right] \times \left[-e^{-qx^{-p}}.log\left(1-e^{-qx^{-p}}\right)\right]^{b-1} e^{\left(-a\left[-e^{-qx^{-p}}.log\left(1-e^{-qx^{-p}}\right)\right]^{b}\right)} \tag{6}$$

In order to integrate the HWIW distribution with N.L, the random variable and parameters of HWIW distribution are converted to Neutrosophic random variable and Neutrosophic parameters as follows:

Let $X_N = d + tI,\ tI \in [X_L, X_U]$, where $X_L,\ X_U$ are lower and upper values of the neutrosophic random variable having determined part d and indeterminate part tI, $tI \in [I_L, I_U]$. Note that the NHWIW distribution reduces to classical HWIW distribution when $X_L = X_U$. The neutrosophic cumulative density function (NCDF) of NHWIW has a Neutrosophic shape parameters $a_N \in [a_L, a_U]$, $b_N \in [b_L, b_U]$, $q_N \in [q_L, q_U]$, and $p_N \in [p_L, p_U]$, has the form:

$$F(x_N) = 1 - e^{\left(-a_N\left[-e^{-q_N x_N^{-p_N}}.log\left(1-e^{-q_N x_N^{-p_N}}\right)\right]^{b_N}\right)}, \quad x_N\ a_N, b_N, q_N, p_N > 0 \tag{7}$$

To obtain the nature of NCDF, the function is plotted with different intervals of parameters and in 2-dimentional and 3-dimensional forms as follows:

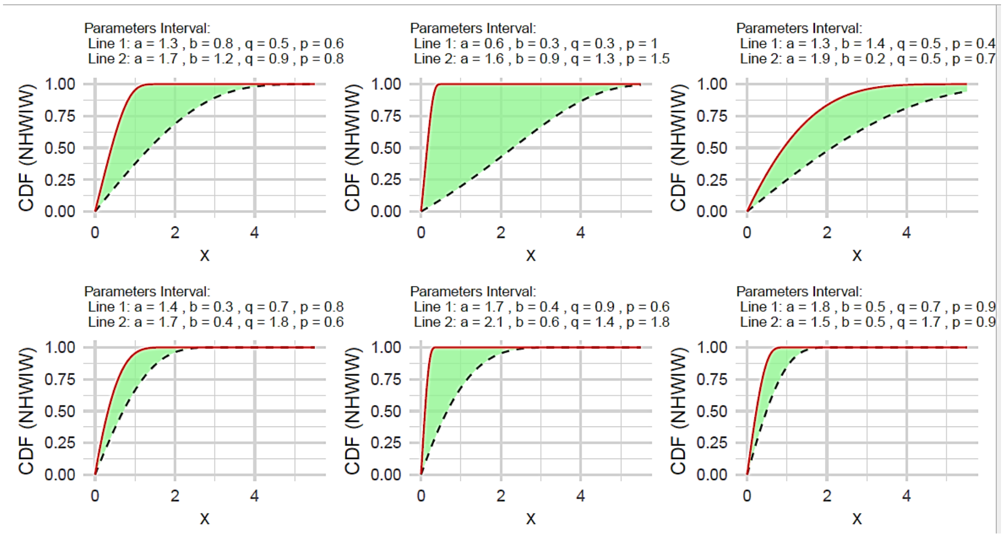


Figure 1. plot of NCDF for NHWIW distribution

And the probability density function (NCDF) of NHWIW has a form:

$$f(x_N) = q_N p_N x_N^{-(p_N+1)} e^{-q_N x_N^{-p_N}} \left[\frac{e^{-q_N x_N^{-p_N}}}{1 - e^{-q_N x_N^{-p_N}}} - log\left(1 - e^{-q_N x_N^{-p_N}}\right) \right] \times \left[-e^{-q_N x_N^{-p_N}} . log\left(1 - e^{-q_N x_N^{-p_N}}\right) \right]^{b_N - 1} e^{\left(-a_N \left[-e^{-q_N x_N^{-p_N}} . log\left(1 - e^{-q_N x_N^{-p_N}}\right) \right]^{b_N} \right)} \quad (8)$$

To obtain the nature of NCDF, the function is plotted with different intervals of parameters and in 2-dimentional and 3-dimensional forms as follows:

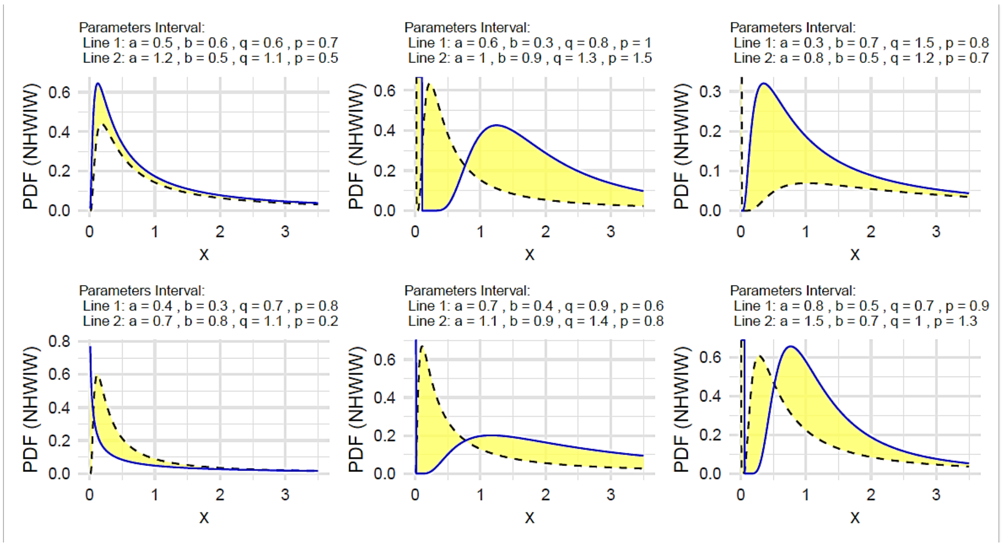


Figure 2. plot of NPDF for NHWIW distribution

While the survival function foe NHWIW distribution given by formula [12]:

$$S(x_N) = e^{\left(-a_N\left[-e^{-q_N x_N^{-p_N}}.log\left(1-e^{-q_N x_N^{-p_N}}\right)\right]^{b_N}\right)} \tag{9}$$

And the hazard function has a formula:

$$h(x_N) = \frac{q_N p_N x_N^{-(p_N+1)} e^{-q_N x_N^{-p_N}} \left[\frac{e^{-q_N x_N^{-p_N}}}{1-e^{-q_N x_N^{-p_N}}} - log\left(1-e^{-q_N x_N^{-p_N}}\right)\right]}{\left[-e^{-q_N x_N^{-p_N}}.log\left(1-e^{-q_N x_N^{-p_N}}\right)\right]^{1-b_N}} \tag{10}$$

3. Properties for NHWIW distribution

In this section we will prove some statistical properties for NHWIW distribution, and show what changing of classical IW distribution.

3.1 NCDF and NPDF expansion

Due to the difficulty of NCDF and NPDF functions in equations (7) and (8) respectively, these functions are simplified in order to simplify the proof of NHWIW distribution properties. This is done using binomial expansion, the exponential function expansion, and the logarithm expansion. Therefore, the simplified NCDF function is obtained as follows:

$$F(x_N) = 1 - \psi\left(e^{-q_N x_N^{-p_N}}\right)^{j+2ib_N} \tag{11}$$

Where $\psi = \sum_{i=j=0}^{\infty} \frac{(-1)^{i+ib_N+j}}{i!} {a_N}^{i} d_{ib_N,j}$, and $d_{ib_N,j} = j^{-1}\sum_{m=1}^{j} \frac{m(ib_N+1)-j}{m+1}\quad for\ j \geq 0\ and\ d_{ib_N,0} = 1$

By same way we can expansion NPDF function to get a form:

$$f(x_N) = \mathrm{M}x_N^{-(p_N+1)} e^{-(2ib_N+2b_N+j+k)q_N x_N^{-p_N}} - N x_N^{-(p_N+1)} e^{-(2ib_N+2b_N+j+z)q_N x_N^{-p_N}} \tag{12}$$

Where $M = \sum_{i=j=k=0}^{\infty} \frac{(-1)^{i+ib_N+b_N-1+j+k}}{i!} {a_N}^{i+1} d_{ib_N+b_N-1,j}\, b_N\, q_N p_N$

And $N = \sum_{i=j=z=0}^{\infty} \frac{(-1)^{i+ib_N+b_N-1+j}}{i!} {a_N}^{i+1} d_{ib_N+b_N-1,j} d_{1,z}\, b_N\, q_N p_N$

As $d_{1,z} = z^{-1}\sum_{m=1}^{j} \frac{2m-j}{m+1}\quad for\ z \geq 0\quad and\ d_{1,0} = 1$ and $d_{ib_N+b_N-1,j} = j^{-1}\sum_{m=1}^{j} \frac{m(ib_N+b_N)-j}{m+1}\quad for\ j \geq 0\quad and\ d_{ib_N+b_N-1,0} = 1$

3.2 Moment

Let X_N be any Neutrosophic random variable, then the n^{th} moment for NHWIW distribution is given by form [13], [14], [15], [16]

$$\mu_n = E(x_N^n)_{NHWIW} = \int_{-\infty}^{\infty} x_N^n f(x_N) dx_N \tag{13}$$

By putting equation (12) in equation (13) we have got a form:

$$\mu_n = \mathrm{M}\int_0^{\infty} x_N^{n-(p_N+1)} e^{-(2ib_N+2b_N+j+k)q_N x_N^{-p_N}} - N\int_0^{\infty} x_N^{n-(p_N+1)} e^{-(2ib_N+2b_N+j+z)q_N x_N^{-p_N}}\, dx_N$$

To get a final form:

$$\mu_n = \frac{\Gamma\left(\frac{n-p_N}{p_N}\right)}{p_N q_N^{\frac{n-p_N}{p_N}}}\left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{n-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{n-p_N}{p_N}}}\right] \tag{14}$$

The first four moments are found by substituting the value of n and as follows:

$$\mu_1 = \frac{\Gamma(\frac{1-p_N}{p_N})}{p_N q_N^{\frac{1-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{1-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{1-p_N}{p_N}}} \right] \quad (15)$$

$$\mu_2 = \frac{\Gamma(\frac{2-p_N}{p_N})}{p_N q_N^{\frac{2-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{2-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{2-p_N}{p_N}}} \right] \quad (16)$$

$$\mu_3 = \frac{\Gamma(\frac{3-p_N}{p_N})}{p_N q_N^{\frac{3-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{3-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{3-p_N}{p_N}}} \right] \quad (17)$$

$$\mu_4 = \frac{\Gamma(\frac{4-p_N}{p_N})}{p_N q_N^{\frac{4-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{4-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{4-p_N}{p_N}}} \right] \quad (18)$$

From it, we can obtain the skewness and Kurtosis of NHWIW distribution respectively as follows [16]:

$$SK_{NHWIW} = \frac{\frac{\Gamma(\frac{3-p_N}{p_N})}{p_N q_N^{\frac{3-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{3-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{3-p_N}{p_N}}} \right]}{\left(\frac{\Gamma(\frac{2-p_N}{p_N})}{p_N q_N^{\frac{2-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{2-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{2-p_N}{p_N}}} \right] \right)^{\frac{3}{2}}} \quad (19)$$

$$KU_{NHWIW} = \frac{\frac{\Gamma\left(\frac{4-p_N}{p_N}\right)}{p_N q_N^{\frac{4-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{4-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{4-p_N}{p_N}}} \right]}{\left(\frac{\Gamma\left(\frac{2-p_N}{p_N}\right)}{p_N q_N^{\frac{2-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{2-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{2-p_N}{p_N}}} \right] \right)^{2}} - 3 \quad (20)$$

To know the change in the moments of NHWIW distribution with change in intervals of Neutrosophic parameters, table 1 shows a set of moments with variance, skewness, and Kurtosis as follows:

Table.***1*** some intervals of moments for NHWIW

a_N	b_N	q_N	p_N	$\grave{\mu}_{1_N}$	$\grave{\mu}_{2_N}$	$\grave{\mu}_{3_N}$	$\grave{\mu}_{4_N}$	σ_N^2	S_N	K_N
[1.8,2.8]	[1.6,2.6]	[1.4, 2.4]	[1.1,2.1]	[2.146064, 2.857927]	[4.723157, 10.57421]	[10.64869, 52.02946]	[24.5692,3 54.4007]	[0.117566, 2.406463]	[1.037403, 1.513134]	[1.101352, 3.169558]
			[1.2,2.2]	[2.0717, 2.593549]	[4.391856, 8.343002]	[9.518114, 33.94803]	[21.06914, 179.7407]	[0.099915, 1.616506]	[1.034139, 1.408741]	[1.092322, 2.58227]
		[1.6, 2.6]	[1.3,2.3]	[2.077192, 2.651157]	[4.406707, 8.434903]	[9.540035, 32.68622]	[21.05919, 157.4864]	[0.09198,1 .40627]	[1.031285, 1.334273]	[1.084459, 2.213517]
			[1.4,2.4]	[2.014016, 2.458261]	[4.135742, 7.066473]	[8.652692, 24.02667]	[18.43118, 98.13544]	[0.079482, 1.023426]	[1.028777, 1.279056]	[1.077572, 1.965262]

	[1.9,2.9]	[1.8, 2.8]	[1.5,2.5]	[2.047589, 2.552581]	[4.257436, 7.198049]	[8.983496, 22.42205]	[19.22557, 77.17473]	[0.064815, 0.682379]	[1.022641, 1.161056]	[1.060677, 1.489517]
			[1.6,2.6]	[1.991347, 2.400354]	[4.022199, 6.28886]	[8.235853, 17.98276]	[17.08652, 56.13042]	[0.056736, 0.527161]	[1.020971, 1.140246]	[1.056152, 1.419235]
		[1.9,2.9]	[1.7,2.7]	[1.966076, 2.347823]	[3.916789, 5.956809]	[7.902665, 16.33094]	[16.14081, 48.38371]	[0.051334, 0.444536]	[1.019479, 1.123287]	[1.052119, 1.363552]
			[1.8,2.8]	[1.894836, 2.153215]	[3.634783, 5.001715]	[7.055468, 12.54589]	[13.85253, 34.02125]	[0.04438,0 .36538]	[1.018141, 1.121561]	[1.048509, 1.359917]

Table 1 presents the different values of statistical moments (mean, variance, skewness, kurtosis) for the NHWIW distribution with changes in the upper and lower boundary parameters of the unspecified variables. The values show that the moments of the distribution change significantly when the boundaries are modified. Skewness and kurtosis are used to assess the shape of the distribution. The mean expresses the expected center of the distribution and changes with the variation of the ambiguous parameters. The variance. High values indicate that the distribution can handle disparate and scattered data, while it indicates the level of concentration in the values. Low values indicate that the distribution can handle data with a wide range. Low values indicate that the distribution is balanced, which enhances its suitability to realistic data.

3.3 Moment Generating Function

Let X_N be any Neutrosophic random variable, then the Moment Generating Function (MGF) for NHWIW distribution is given by form [17] , [18] :

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x_N) dx$$

From equation (14) and using exponential expansion we get a final form:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[\frac{\Gamma(\frac{n-p_N}{p_N})}{p_N q_N^{\frac{n-p_N}{p_N}}} \left[\frac{M}{(2ib_N+2b_N+j+k)^{\frac{n-p_N}{p_N}}} - \frac{N}{(2ib_N+2b_N+j+z)^{\frac{n-p_N}{p_N}}} \right] \right] \qquad (21)$$

3.4 Quantile function of NHWIW distribution

The Quantile function has a major role in application of Monte Carlo simulation and represents the inverse of NCDF function $Q(u) = F^{-1}(u)$ [19], which is obtained for the NHWIW distribution as follows:

$$Q(u) = \left[\frac{\log\left[\frac{\Theta}{\Theta+W_{-1}(\Theta)e^{\Theta}}\right]}{q_N} \right]^{\frac{-1}{b_N}}, \Theta = \left[\frac{-\log(1-u)}{a_N}\right]^{\frac{1}{b_N}} \qquad (22)$$

Table 2 expresses a set of intervals of the Quintile function values of NHWIW for different intervals as follow:

Table 2: Quintile function values of NHWIW for different intervals

s_N	(a_N, b_N, q_N, p_N)				
	[0.4, 0.8],[1.1, 1.9] [0.4, 1.4],[1.2, 1.7]	[0.6, 1.6],[1, 1.5] [0.4, 1.4],[1, 1.5]	[0.5,1.5],[1.1.1,6] [0.51.5],[1.1,1.6]	[1.2,1.7],[1,1.5] [0.8,1.8],[1.2,1.7]	[1.1,1.7],[1.9,1.9] [0.7,1.7],[1.5,2]
0.1	[0.5880204,1.50829]	[0.407121, 1.194348]	[0.622559, 1.33252]	[0.669702, 3.3304]	[0.9420654, 3.28916]
0.2	[0.8511505,1.75326]	[0.6020268, 1.40746]	[0.898051, 1.54565]	[0.86103, 1.38635]	[1.1027903, 1.56190]
0.3	[1.1616532, 1.95887]	[0.8308531, 1.59006]	[1.214072, 1.72384]	[1.047516, 1.5928]	[1.2370894, 1.68682]

0.4	[1.5748618, 2.15671]	[1.1324191, 1.76916]	[1.621220, 1.89484]	[1.253187, 1.7644]	[1.3657090, 1.80121]
0.5	[2.1794962, 2.36248]	[1.5680264, 1.95911]	[2.07237,2.194616]	[1.499006, 1.9283]	[1.4988826, 1.91481]
0.6	[2.590992,3.165274]	[2.174631,2.267291]	[2.26927,3.086756]	[1.816802, 2.0979]	[1.6459544, 2.03529]
0.7	[2.863979,5.033473]	[2.43857,3.5705340]	[2.50435,4.680039]	[2.270645, 2.2853]	[1.8204959, 2.17226]
0.8	[3.226530,9.576866]	[2.800418,6.695981]	[2.816750,8.25737]	[2.50855,3.02963]	[2.0505678, 2.34410]
0.9	[3.821119,28.62166]	[3.423055,19.87242]	[3.33042,21.51546]	[2.80425,4.80523]	[2.4236659, 2.60524]

Table 2 presents the values of the quantile function for the NHWIW distribution for different probability intervals. The different values reflect changes in the range of the data distribution based on the probability. Narrower intervals show greater accuracy in representing the data, while wider intervals reflect greater flexibility. Small values indicate the distribution's ability to provide accurate boundaries for the intervals.

3.5 Renyi entropy

The Renyi entropy is given by form [20], [21] :

$$I_R(c)_{NHWIW} = \frac{1}{1-c} log \int_0^\infty f(x_N)^c dx$$

$$I_R(c)_{NHWIW} = \frac{1}{1-c} log \int_0^\infty (M x_N^{-(p_N+1)} e^{-(2ib_N+2b_N+j+k)q_N x_N^{-p_N}} - N x_N^{-(p_N+1)} e^{-(2ib_N+2b_N+j+z)q_N x_N^{-p_N}})^c dx$$

Then the final form:

$$I_R(c)_{NHWIW} = \frac{1}{1-c} \log\left[R.\Gamma\left(\frac{1-c(p_N+1)}{p_N}\right)\right]$$

$$R = \sum_{n=0}^{c} (-1)^n \binom{c}{n} M^{c-n} N^n \frac{1}{p_N q^{\frac{-c(p_N+1)}{p_N}+\frac{1}{p_N}} (2icb_N + 2cb_N + cj + ck - nk + nz)^{\frac{-c(p_N+1)}{p_N}+\frac{1}{p_N}}} \tag{23}$$

4. Estimation

4.1 maximum likelihood estimation

The NHWIW distribution parameters are determine using maximum likelihood estimation approach. For sample $x_{N_1}, x_{N_2}, \dots, x_{N_m}$ the random sample [22], [23], [24], [25], [26] The NHWIW distribution NPDF is followed:

$$L(\theta_N, x_{N_i}) = \prod_{i=1}^{m} q_N p_N x_{N_i}^{-(p_N+1)} e^{-q_N x_{N_i}^{-p_N}} \left[\frac{e^{-q_N x_{N_i}^{-p_N}}}{1-e^{-q_N x_{N_i}^{-p_N}}} - log\left(1 - e^{-q_N x_{N_i}^{-p_N}}\right)\right]$$

$$\times \left[-e^{-q_N x_{N_i}^{-p_N}} . log\left(1 - e^{-q_N x_{N_i}^{-p_N}}\right)\right]^{b_N - 1} e^{\left(-a_N\left[-e^{-q_N x_{N_i}^{-p_N}} .log\left(1-e^{-q_N x_{N_i}^{-p_N}}\right)\right]^{b_N}\right)}$$

we compute the log- likelihood:

$$L = mlog(q_N) + mlog(p_N) - (p_N - 1)\sum_{i=1}^{m} \log(x_{N_i}) - \sum_{i=1}^{m} q_N x_{N_i}^{-p_N} \tag{24}$$

$$+\sum_{i=1}^{m}\log\left[\frac{e^{-q_N x_{N_i}^{-p_N}}}{1-e^{-q_N x_{N_i}^{-p_N}}}-log\left(1-e^{-q_N x_{N_i}^{-p_N}}\right)\right]$$
$$+(b_N-1)\sum_{i=1}^{m}\log\left[-e^{-q_N x_{N_i}^{-p_N}}.log\left(1-e^{-q_N x_{N_i}^{-p_N}}\right)\right]$$
$$-a_N\sum_{i=1}^{m}\left[-e^{-q_N x_{N_i}^{-p_N}}.log\left(1-e^{-q_N x_{N_i}^{-p_N}}\right)\right]^{b_N}$$

4.2 Least square estimation

The following formula can be used to estimate a parameters using the Least square estimation (LSE) method [27], [28]:

$$\varphi(\theta_N)=\sum_{i=1}^{m}\left[1-e^{\left(-a_N\left[-e^{-q_N x_{N_i}^{-p_N}}.log\left(1-e^{-q_N x_{N_i}^{-p_N}}\right)\right]^{b_N}\right)}-\frac{1}{n+1}\right]^2 \quad (25)$$

4.3 Weighted Least square estimation

The following formula can be used to estimate a parameters using the Weighted Least square estimation (WLSE) method [29]:

$$V(\theta_N)=\sum_{i=1}^{m}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-e^{\left(-a_N\left[-e^{-q_N x_{N_i}^{-p_N}}.log\left(1-e^{-q_N x_{N_i}^{-p_N}}\right)\right]^{b_N}\right)}-\frac{i}{n+1}\right]^2 \quad (26)$$

Estimates of the parameters for the three previously described methods may be obtained by finding the partial derivative of four parameters and setting it to zero. Computer technologies such as the R language are used since it is difficult to find these values in numerical solutions.

5. Simulation

To demonstrate the efficieny of the estimation of NHWIW distribution, a Monte Carlo simulation is conducted for the three methods presented in a fourth section, where the sizes of generated samples were relied upon at n=50, 100, 150, and 200, to 1000 with the calculation of the values of mean square error (MSE), and its root (RMSE) [30], and the calculation of the bias in the estimated parameters, where Table 3 shows the simulation values as follows:

Table 3 : Monte Carlo simulations conducted for the NHWIW

$\boldsymbol{r_N}$ = [0.4,1.4], $\boldsymbol{u_N}$ = [0.5,1.5], $\boldsymbol{b_N}$ = [0.7,1.7], $\boldsymbol{c_N}$ = [0.8,1.8]					
N	**Est.**	**Ess. Par.**	**MLE**	**LSE**	**WLSE**
50	**Mean**	$\widehat{a_N}$	[2.71415, 3.8807]	[1.81833, 2.2308]	[2.18454, 2.62480]

		$\widehat{b_N}$	[2.14886, 3.4619]	[2.00585, 2.14894]	[2.28999, 2.41772]
		$\widehat{q_N}$	[1.98333, 2.030004]	[1.68494, 2.65528]	[1.49501, 2.36023]
		$\widehat{p_N}$	[1.78887, 2.01288]	[1.74966, 2.63342]	[1.46794, 2.40224]
	MSE	$\widehat{a_N}$	[22.2105,25.1223]	[1.40892, 1.91622]	[2.77863, 3.67925]
		$\widehat{b_N}$	[2.20026, 5.72680]	[0.44948,1.0952]	[0.98082,1.91795]
		$\widehat{q_N}$	[3.88680, 5.14601]	[0.96725, 2.38973]	[1.04593, 2.03972]
		$\widehat{p_N}$	[1.26702, 2.03484]	[0.96801, 1.11202]	[0.64689, 0.95739]
	RMSE	$\widehat{a_N}$	[4.7128,5.01222]	[1.18698, 1.38427]	[1.66692, 1.91813]
		$\widehat{b_N}$	[1.48332, 2.3930]	[0.67043,1.04654]	[0.99036,1.38490]
		$\widehat{q_N}$	[1.97149, 2.26848]	[0.98348, 1.54587]	[1.02270, 1.42818]
		$\widehat{p_N}$	[1.12562, 1.42648]	[0.98387, 1.05452]	[0.80430, 0.97846]
	Bias	$\widehat{a_N}$	[1.314151, 1.9807]	[0.33082,0.41833]	[0.72480,0.78454]
		$\widehat{b_N}$	[0.64886, 1.4619]	[0.14894,0.50585]	[0.41772,0.78999]
		$\widehat{q_N}$	[0.26999,0.28333]	[0.015057, 0.35528]	[0.06023,0.20498]
		$\widehat{p_N}$	[0.01112, 0.48711]	[0.05033, 0.13342]	[0.33205, 0.09775]
100	**Mean**	$\widehat{a_N}$	[2.31534, 3.4201]	[1.85065, 2.25892]	[1.96960, 2.52434]
		$\widehat{b_N}$	[1.6431, 2.78082]	[1.95885, 2.19012]	[1.95962, 2.37243]
		$\widehat{q_N}$	[2.04081,2.10196]	[1.54758, 2.51858]	[1.44971, 2.36524]
		$\widehat{p_N}$	[1.80824, 2.09797]	[1.62019, 2.56520]	[1.48308, 2.427468]
	MSE	$\widehat{a_N}$	[6.10985, 16.8563]	[1.28531, 1.92259]	[1.61472, 2.84221]
		$\widehat{b_N}$	[0.52321, 2.10325]	[0.42942,0.75540]	[0.67244, 0.73143]
		$\widehat{q_N}$	[1.90643, 2.19297]	[0.51869, 1.71932]	[0.362008, 1.65003]
		$\widehat{p_N}$	[0.70767,0.72883]	[0.52677, 0.82273]	[0.32772, 0.84436]
	RMSE	$\widehat{a_N}$	[2.47181, 4.10565]	[1.13371, 1.38657]	[1.27071, 1.68588]
		$\widehat{b_N}$	[0.72333, 1.45026]	[0.65530,0.86914]	[0.82002, 0.85523]
		$\widehat{q_N}$	[1.38073, 1.48086]	[0.72020, 1.31123]	[0.60167, 1.28453]
		$\widehat{p_N}$	[0.84123,0.85372]	[0.72579, 0.90704]	[0.57247, 0.91889]
	Bias	$\widehat{a_N}$	[0.91534, 1.52011]	[0.35892,0.45065]	[0.56960, 0.62434]
		$\widehat{b_N}$	[0.14319, 0.78082]	[0.19012,0.45885]	[0.37243,0.45962]
		$\widehat{q_N}$	[0.25918,0.40196]	[0.15241, 0.21858]	[0.06524,0.25028]

		$\widehat{p_N}$	[0.00824, 0.40202]	[0.065207,0.17980]	[0.07253,0.31691]
150	Mean	$\widehat{a_N}$	[2.39295, 2.59663]	[1.69799, 2.28928]	[1.84874, 2.51742]
		$\widehat{b_N}$	[1.48475, 2.38927]	[1.84251, 2.22170]	[1.82141, 2.36527]
		$\widehat{q_N}$	[2.32242, 2.26305]	[1.56125, 2.38488]	[1.55411, 2.22098]
		$\widehat{p_N}$	[1.83872, 2.33146]	[1.62906, 2.47499]	[1.61889, 2.31876]
	MSE	$\widehat{a_N}$	[5.19190,6.34498]	[0.76663, 1.61336]	[1.38675, 2.57476]
		$\widehat{b_N}$	[0.36447, 0.76441]	[0.41032,0.45602]	[0.17093, 0.63993]
		$\widehat{q_N}$	[1.85714,2.85060]	[0.39942, 0.41032]	[0.39838, 1.05506]
		$\widehat{p_N}$	[0.30307, 0.49069]	[0.29310, 1.01496]	[0.40961, 0.52287]
	RMSE	$\widehat{a_N}$	[2.27857,2.51892]	[0.87557, 1.27018]	[0.27180, 1.60460]
		$\widehat{b_N}$	[0.60371, 0.8743]	[0.64056,0.67529]	[0.63118, 0.79996]
		$\widehat{q_N}$	[1.36277,1.68837]	[0.63200, 1.00745]	[0.64001, 1.02716]
		$\widehat{p_N}$	[0.55052, 0.70049]	[0.54139, 0.75893]	[0.52135, 0.72310]
	Bias	$\widehat{a_N}$	[0.69663,0.99295]	[0.38928,1.17760]	[0.44874, 0.61742]
		$\widehat{b_N}$	[0.015245, 0.38927]	[0.22170,0.29799]	[0.32141, 0.36527]
		$\widehat{q_N}$	[0.036943,0.62242]	[0.08488,0.34251]	[0.07901,0.14588]
		$\widehat{p_N}$	[0.03872, 0.16853]	[0.025003,0.13874]	[0.18110, 0.18123]
200	Mean	$\widehat{a_N}$	[2.56066, 2.69271]	[1.65731, 2.22683]	[1.74361, 2.39868]
		$\widehat{b_N}$	[1.428845, 2.33717]	[1.77586, 2.16486]	[1.70362, 2.29352]
		$\widehat{q_N}$	[2.19764,2.30402]	[1.61052, 2.37936]	[1.65621, 2.25097]
		$\widehat{p_N}$	[1.77657, 2.30401]	[1.68037, 2.492011]	[1.66439, 2.36508]
	MSE	$\widehat{a_N}$	[6.98059,8.23299]	[0.73184, 1.36073]	[0.941745, 2.21622]
		$\widehat{b_N}$	[0.25652, 0.56396]	[0.31173,0.35832]	[0.28724, 0.46309]
		$\widehat{q_N}$	[1.12979,2.25599]	[0.351277, 0.86185]	[0.39563, 1.00733]
		$\widehat{p_N}$	[0.15658, 0.33635]	[0.29343, 0.49330]	[0.19297, 0.43221]
	RMSE	$\widehat{a_N}$	[2.64208,2.86931]	[0.85547, 1.16650]	[0.97043, 1.48869]
		$\widehat{b_N}$	[0.506478, 0.75097]	[0.55833,0.59860]	[0.53595, 0.68050]
		$\widehat{q_N}$	[1.06291,1.50199]	[0.59268, 0.92836]	[0.62899, 1.00366]
		$\widehat{p_N}$	[0.39571, 0.57996]	[0.54169, 0.70235]	[0.43929, 0.65742]
	Bias	$\widehat{a_N}$	[0.79271,1.16066]	[0.25731, 0.32683]	[0.34361, 0.49868]

		$\widehat{b_N}$	[0.071154, 0.33717]	[0.16486,0.27586]	[0.20362, 0.29352]
		$\widehat{q_N}$	[0.10235,0.60402]	[0.07936,0.08947]	[0.043784, 0.04902]
		$\widehat{p_N}$	[0.023423, 0.19598]	[0.00798,0.11962]	[0.13491,0.13560]

Table 3 shows the lowest MSE and RMSE values for the MLE method, indicating that it is the most accurate. Bias is low in all methods, reflecting the quality of the estimates. MLE (Maximum Likelihood Estimation) showed superior performance, especially with large sample sizes, making it the best fit for the distribution. LSE and WLSE perform well with small sample sizes but are less accurate as the size increases.

6. Application

To demonstrate the extent of the quality of the distribution and its efficiency in practical applications, the practical aspect is an important aspect to show this, as in this part a practical applications is conducted on real neutrosophic data that is used [31], with a comparison of the results obtained between the proposed distribution and six other distribution represented by:

- Neutrosophic beta inverse Wiebull (NBeIW)
- Neutrosophic Kumaraswamy inverse Wiebull (NKuIW)
- Neutrosophic Exponeted generalized inverse Wiebull (NEGIW)
- Neutrosophic log-Gamma inverse Wiebull (NLGamIW)
- Neutrosophic Gompertz inverse Wiebull (NGoIW)
- Neutrosophic inverse Wiebull (NIW)

This comparision required the use of four information criteria, which are (AIC [19], CAIC [32], [33], HQIC [34], [35], , and BIC [36]) in addition to four statistical measures, which are (Kolmogorov-Smirnov (KS), Anderson- Darling (A), Cramér-von Mises (W), and p-value [37], [38]).

Data set-1

The first represented by mortality data for children under five years of age [31]**.**

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	**26**	**[15.76, 16.76]**	**[7.27, 7.37]**	**[14.21, 15.32]**	**[15.22, 16.26]**	**[7.87, 8.27]**	**[6.81, 7.98]**	**[31.53, 31.81]**	**[23.83, 24.72]**	**0.56**	**[-0.96, -0.93]**	**[1.43, 1.45]**

The results of the criteria for the distributions were displayed in Table 4, while Table 5 expressed the value of the statistical measures, while Table 6 displayed the values of the estimated parameters for each distribution.

Table 4. results of the criteria for the distributions

Dist.	-2L	AIC	CAIC	BIC	HQIC
HWIW	[85.40969,85.5326]	[178.8194,179.06]	[180.7241,180.97]	[183.851,184.097]	[180.268,180.514]
BeIW	[86.0416, 87.6206]	[180.089, 183.302]	[181.994, 185.207]	[185.122, 188.334]	[181.538, 184.751]
KuIW	[86.1110, 86.5381]	[180.234, 181.085]	[182.139, 182.990]	[185.267, 186.118]	[181.683, 182.535]
EGIW	[86.2713,86.31197]	[180.564,180.649]	[182.469,182.554]	[185.597,185.681]	[182.014,182.098]
LGamIW	[85.63031,85.8919]	[179.260,179.785]	[181.165,181.690]	[184.293,184.818]	[180.709,181.235]
GoIW	[87.0561, 87.2476]	[182.112, 182.495]	[184.017, 184.400]	[187.144, 187.527]	[183.561, 183.944]
IW	[96.1148, 104.916]	[196.24, 213.9324]	[196.761, 214.454]	[198.756, 216.448]	[196.9645, 214.65]

From Table 4, NHWIW achieved the lowest values for most of the criteria, indicating its high fit to the data. The IW distribution was the least efficient due to its high values, as low values of the criteria enhance the stability of the distribution when comparing models with different complexities .

Table 5. value of the statistical measures

Dist.	W	A	K-S	p-value
HWIW	[0.03774283,0.0473640]	[0.313303,0.341763]	[0.103911, 0.106321]	[0.900544,0.9142441]
BeIW	[0.04349193,0.0582771]	[0.3464307,0.40208]	[0.088246, 0.119375]	[0.8107124,0.9765447]
KuIW	[0.04234476,0.0551776]	[0.341253,0.386771]	[0.081400, 0.094516]	[0.9572919,0.9896749]
EGIW	[0.04774075,0.0614708]	[0.370513,0.420625]	[0.082911, 0.090155]	[0.9714649,0.9874063]

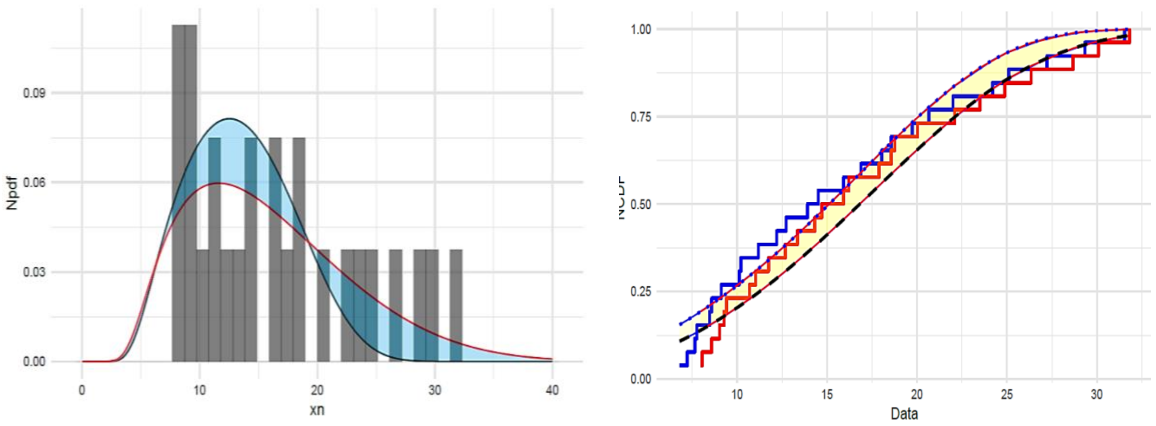
LGamIW	[9.373411,9.426508]	[52.44084, 52.46839]	[0.9483765, 0.94545]	5.551115e-16
GoIW	[0.0627775,0.06469447]	[0.461921, 0.464890]	[0.12449, 0.1460138]	[0.5858848,0.7699224]
IW	[0.04225913,0.0575322]	[0.339643,0.397695]	[0.227421, 0.364774]	[0.001313552,0.1155558]

Table 5 shows that NHWIW achieved the highest p-value and lowest W and A values, reflecting its high fit to the data. Other distributions showed poorer performance compared to NHWIW .

Table 6. Estimator value interval for parameters by MLE

Dist.	$\hat{a}_N$	$\hat{b}_N$	$\hat{q}_N$	$\hat{p}_N$
HWIW	[0.013462, 0.0280360]	[3.3452479,3.5492839]	[7.5562016, 12.242277]	[1.918304,1.948398]
BeIW	[5.941547,6.880572]	[3.479567, 3.568686]	[6.136480,7.531162]	[0.966692,1.129877]
KuIW	[5.85491, 5.858095]	[4.048258,4.141243]	[6.026946, 6.382458]	[1.111050,1.119379]
EGIW	[3.8376168, 4.5794689]	[6.6182216, 7.6331922]	[7.205414,7.2723604]	[0.789913,0.848330]
LGamIW	[7.337486, 8.415236]	[5.380997, 6.665243]	[7.4194373, 8.495675]	[1.17855,1.194137]
GoIW	[0.006215729,0.00781413]	[1.4543039, 1.4894439]	[1.9634420,1.9943284]	[1.486703,1.493761]
IW	---	---	[6.1610960,13.242297]	[0.8713852,1.14196]

Table 6 shows the estimated intervals of the main parameters of the NHWIW distribution compared to other distributions (such as BeIW, KuIW, etc.) using the maximum likelihood estimation (MLE) method for the first dataset. The NHWIW distribution showed narrow intervals of the parameters, indicating the stability of the estimates, while other distributions such as BeIW and KuIW showed greater variation in the intervals of the parameters, which may indicate their poor efficiency compared to NHWIW.



(a) (b)

Figure 3: (a) Fitting pdfs NHWIW with histogram data set, (b) Empirical Fitted CDFs NHWIW with data set

Figure 3 (a) shows how well the probability density function (PDF) of the NHWIW distribution matches the actual data for the first set. The distribution shows a strong fit with the structural distribution of the data, with the curve following the shape of the experimental data. This indicates the ability of NHWIW to accurately and flexibly represent real data. Figure 3 (b) shows the empirical CDF compared with the theoretical CDF. The strong fit between the distributions reflects the efficiency of NHWIW in representing the cumulative probability of the data.

Data set-2

The second is COVID-19 in Netherlands for thirty days [39]

Var	N	Mean	SD	Median	Trimmed	Mad	Min	Max	Range	SK	KU	Se
1	30	[6.14, 6.36]	[3.51, 3.56]	[5.37, 5.64]	[5.79, 6]	[2.72, 2.85]	[1.27, 1.34]	[14.92, 15.66]	[13.64, 14.33]	[0.8, 0.82]	[-0.18, 0.03]	[0.64, 0.65]

Table 7. results of the criteria for the distributions

Dist.	-2L	AIC	CAIC	BIC	HQIC
HWIW	[76.43589,77.09943]	[160.8718, 162.1989]	[162.4718,163.7989]	[166.4766,167.8036]	[162.6648,163.9919]
BeIW	[76.48142,77.32048]	[160.9628, 162.641]	[162.5628,164.241]	[166.5676,168.2457]	[162.7559,164.434]
KuIW	[76.59179, 77.27442]	[161.1836, 162.5488]	[162.7836,164.1488]	[166.7884,168.1536]	[162.9766,164.3419]
EGIW	[78.25885, 79.37512]	[164.5248, 166.7519]	[166.1248,168.3519]	[170.1296,172.3567]	[166.3179,168.545]
LGamIW	[76.56406, 77.31349]	[161.1281, 162.627]	[162.7281,164.227]	[166.7329,168.2318]	[162.9211,164.42]
GoIW	[77.94244, 79.25779]	[163.9185, 166.5239]	[165.5185,168.1239]	[169.5233,172.1287]	[165.7115,168.3169]
IW	[80.90795, 81.86503]	[165.8159, 167.7301]	[166.2603,168.1745]	[168.6183,170.5325]	[166.7124,168.6266]

Table 7 compares NHWIW with other distributions using informative criteria such as AIC, BIC, CAIC, and HQIC for the second dataset. NHWIW distribution had the lowest values for all criteria compared to other distributions. Other distributions, such as IW and GoIW, showed high values, indicating their poor performance in fitting the data.

Table 8. value of the statistical measures

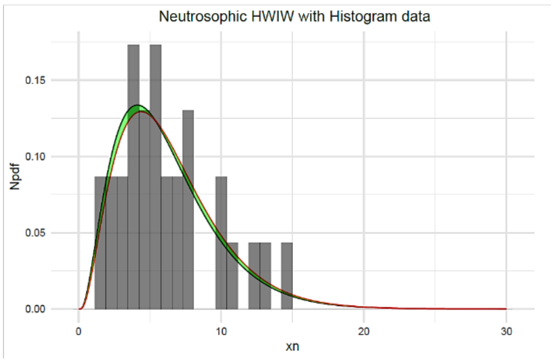
Dist.	W	A	K-S	p-value
HWIW	[0.0233792, 0.0241559]	[0.1664448, 0.1756536]	[0.070432, 0.0770078]	[0.9881955, 0.99588]
BeIW	[0.02499, 0.02863132]	[0.185047, 0.2137449]	[0.08221554,0.085797]	[0.96637, 0.977128]
KuIW	[0.0217852, 0.0257614]	[0.1728076, 0.1938708]	[0.0685944, 0.076441]	[0.9891058,0.9971004]
EGIW	[0.0612384, 0.082723]	[0.4535322, 0.57210]	[0.1106459, 0.114588]	[0.784162,0.8173854]
LGamIW	[9.976712,10.01212]	[60.25735,60.30363]	[0.9854649, 0.987072]	1.221245e-15
GoIW	[0.0474965, 0.0719870]	[0.3674051, 0.5081394]	[0.0990987, 0.105585]	[0.8571036,0.9018158]
IW	[0.1411036, 0.1590753]	[0.940501, 1.032139]	[0.1517456, 0.153257]	[0.4380138,0.4503734]

Table 8 NHWIW had the highest p-value and the lowest values for W, A, and K-S, indicating that this distribution is superior to other distributions and provides an excellent fit to the data and there is no evidence to reject the hypothesis of the distribution. Distributions such as IW showed high values for the criteria, reflecting their poor representation of the data.

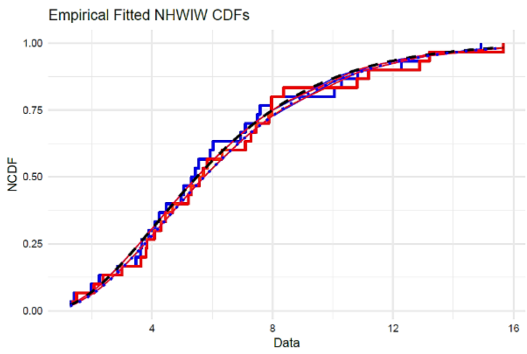
Table 9. Estimator value interval for parameters by MLE

Dist.	$\hat{a}_N$	$\hat{b}_N$	$\hat{q}_N$	$\hat{p}_N$
HWIW	[5.837801, 6.1923656]	[4.02133, 4.2564547]	[0.8979603,0.91083]	[0.3596752,0.37002]
BeIW	[0.326699, 0.5555232]	[23.3992253,26.28143]	[13.330017,19.726555]	[0.642861,0.746774]
KuIW	[3.5882782,9.1795205]	[65.592337, 74.372615]	[1.0179446, 2.7114185]	[0.421744,0.426545]
EGIW	[4.458458,5.3694787]	[2.5368242, 2.864744]	[3.8798198,4.1462443]	[0.652446,0.696044]
LGamIW	[0.4306123, 0.5999024]	[21.714107, 25.562351]	[12.796964,15.607732]	[0.624186,0.698871]
GoIW	[7.8853573,8.9050606]	[0.9466685, 1.2100697]	[7.3119639,8.3552368]	[0.706303,0.712820]
IW	---	---	[7.869781, 8.576315]	[1.553003, 1.564124]

Table 9 NHWIW distribution showed more accurate and stable estimation intervals compared to other distributions. Distributions such as GoIW and BeIW showed wide variations in estimates.



(a)



(b)

Figure 4: (a) Fitting pdfs NHWIW with histogram data set2, (b) Empirical Fitted CDFs NHWIW with data set2

Figure 4 (a) shows how well the PDF of the NHWIW distribution fits the real data of the second group (COVID-19 data). The curve shows that NHWIW captures the main characteristics of the data. The good fit indicates the flexibility of the distribution in dealing with different types of data. Figure 4 (b) compares the empirical CDF with the theoretical NHWIW CDF for the second group of data. The close fit between the two curves reflects the accuracy of NHWIW in predicting the cumulative probabilities of the data. This enhances the reliability of the distribution and its relevance to real-world data such as COVID-19.

Conclusion

The NHWIW distribution showed high efficiency in representing real data compared to 6 other distributions. NHWIW outperformed in information criteria (AIC, BIC, CAIC, HQIC) and statistical criteria (K-S, W, A, p-value). The distribution has great flexibility that enables it to adapt to uncertain or ambiguous data using neutrosophic parameters. The maximum likelihood method (MLE) was the most accurate in estimating the parameters, indicating the stability of the model. Simulations proved that NHWIW provides accurate estimates even with small or medium sample sizes. The distribution is suitable for representing real data such as under-five mortality and COVID-19 data in the Netherlands. NHWIW showed high agreement with experimental data as shown in the figures. NHWIW represents an important step in applying neutrosophic logic in statistical modeling and bridges the research gap in integrating ambiguous data with complex mathematical distributions.

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