Solving System Fredholm Integro-Differential Equation with First and Second Derivatives Using the Successive Approximation Method (SAM)

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Abstract. A system of Fredholm integro-differential equations involving first and second derivatives is addressed using the Successive Approximation Method (SAM) combined with an iterative algorithm. Starting from an initial estimate, the integral term evaluated using previous iterates is treated as a known forcing function in each iteration. This transforms the original integro-differential system into a sequence of linear differential systems, which are then solved under given boundary or initial conditions. Convergence and uniqueness of the solution are rigorously established, and the resulting approximate solution is shown to converge the exact one as the iteration proceeds. Numerical examples demonstrate that SAM produces accurate approximations with controlled error and good computational efficiency.

Keywords: System of Fredholm , integro-differential equations ,The Successive Approximation Method

Introduction

Integral equations can articulate a range of topics in mathematical physics. Certain systems will be utilized as examples; compiling a comprehensive list of such applications would be practically impossible. The body of literature concerning integral equations and their applications is vast. This section will examine systems of Fredholm integro-differential equations of the second kind, expressed as follows:

$$\begin{Bmatrix} \psi^i(x) = f(x) + \lambda_1 \int_a^b [\mu(x,t)\psi(t) + \varepsilon(x,t)\xi(t)]dt \\ \xi^i(x) = g(x) + \lambda_2 \int_a^b [\mu(x,t)\psi(t) + \varepsilon(x,t)\xi(t)]dt \end{Bmatrix} \quad \textbf{(1)}$$

And the kernel of the system form is:

$$\mu(x,t) = \sum_{j=1}^{i-`1} h\mu_{ij}\varphi_i$$

where $\psi^i(x) = \frac{d^i u}{dx^i}$, $\xi^i(x) = \frac{d^i v}{dx^i}$,since the resultant system integrates both the differential and integral operators, it is essential to provide beginning conditions. $\psi(0), \psi'(0), \dots \psi^{(i-1)}(0)$ for the identification of the specific solution $\psi(x,t)$ of the Fredholm integro-differential equation(1).The unknown function $\psi(t), \xi(t)$ that will be established appear within the integral sign, but the derivatives of $\psi(t), \xi(t)$ appear mostly outside the integral sign ,The kernel $\mu(x,t)$and the given $f(x), g(x)$ are real-valued function, and λ_1, λ_2 are arbitrary constants , x is variable and $\psi(t), \xi(t)$, A system of Fredholm integro-differential equations combining first and second derivatives arises in modeling processes with both local dynamics and nonlocal interactions and requires specialized techniques for solution The successive approximation method transforms the original coupled system by isolating the integral terms and treating them as known based on previous iterates in each cycle Through this iterative framework the integro-differential equations are reduced to a sequence of linear ordinary differential problems[1]. each with updated forcing terms determined from the previous iteration The procedure begins with an initial guess that satisfies any prescribed boundary on properties of the kernel functions and the linear operators to ensure that the sequence converges to the true solution which can be rigorously justified under suitable conditions The successive approximation method offers computational efficiency by leveraging existing solvers for ordinary differential equations [2]and[3]. and can achieve high accuracy as iterations progress Numerical case studies demonstrate that SAM effectively handles the coupling between first and second derivatives within the Fredholm framework while maintaining stability and controllable error growth or initial conditions,[4]and[5] and updates the solution by solving the linearized differential systems, then evaluating the integral kernels against the current approximation Convergence of the method[6]and[7] relies thereby providing a robust numerical tool for this class of integro-differential systems.

Methodology Description

The (SAM) gives a system that may solve initial value or integral equations. This approach solves any problems by uncovering consecutive approximations of the answer. The procedure begins with an initial estimate as $p_0(x)$, which is termed, the in approximations and may be any real-valued function $p_0(x)$. The zeroth approximation will then be utilized in a related to recurrence to discover the subsequent approximations. Provide the (NLFIE) of the second kind.

$$\boldsymbol{\psi_n(x) = f(x) + \int_a^b \mu(x,t)\,\psi_{n-1}(t)dt \quad n \geq 1} \qquad \textbf{(2)}$$

where $\psi(x)$ represents the undetermined function that has to be decided and $\mu(x,t)$ represents the kernel.the recurrence connection is shown via the use of the successive approximations method.

$$\boldsymbol{\psi_{j+1}(x) = f(x) + \int_a^b \mu(x,t)\,\psi_j(t)dt \quad , j \geq 0} \qquad \textbf{(3)}$$

Where the zeroth estimate $\psi_0(x)$, might be any specific authentically appreciate function. We always start with a beginning guess for $\psi_0(x)$, and for $\psi_0(x)$, we almost always choose either (0.1) or x as our starting guess. When this value of $\psi_0(x)$ is inputted in to equation (2), some successive estimate of $\psi_j(x), j \geq 1$ will be determined as:

$$\psi_1(x) = f(x) + \int_a^b \mu(x,t)\,\psi_0(t)dt$$

$$\psi_2(x) = f(x) + \int_a^b \mu(x,t)\,\psi_1(t)dt$$

$$\vdots$$

$$\psi_j(x) = f(x) + \int_a^b \mu(x,t)\,\psi_{j-1}(t)dt$$

Consequently, the configuration $\varphi_0(x)$ may be obtained by employing.

$$\psi_0(x) = \lim_{n\to\infty} \psi_{j+1}(x)$$

By way of illustration, the successive approximation technique, also known as the picard iteration strategy, will be discussed in further detail.

Analysis of Fredholm Integro-Differential Equation by Using Successive Approximation Method (SAM)

The Successive Approximation Method is a reliable technique previously utilized to solve Fredholm integro-differential equations. This section will employ cubic splines to address systems of second-kind Fredholm integro-differential equations in a manner consistent with our previous methodology. This technique efficiently resolves any Fredholm problem through Successive Approximation Method, producing accurate answers. This section will analyze systems of Fredholm integro-differential equations as follows**;**

$$\begin{Bmatrix} \boldsymbol{\psi^i(x) = f(x) + \lambda_1 \int_a^b [\mu(x,t)\psi(t) + \mu(x,t)\xi(t)]dt} \\ \boldsymbol{\xi^i(x) = g(x) + \lambda_2 \int_a^b [\mu(x,t)\psi(t) + \mu(x,t)\xi(t)]dt} \end{Bmatrix} \quad (4)$$

And the kernel of the system form is:

$$\mu(x,t) = \sum_{j=1}^{i-`1} h\mu_{ij}\varphi_i$$

Phase1: Integrating both sides the system (1) once or more from 0 to x, utilizing beginning conditions and proceeding accordingly**:**

$$\begin{Bmatrix} \psi(x) = f(x) + \lambda_1 \int_a^b [\mu(x,t)\psi(t) + \mu(x,t)\xi(t)]dt \\ \xi(x) = g(x) + \lambda_2 \int_a^b [\mu(x,t)\psi(t) + \mu(x,t)\xi(t)]dt \end{Bmatrix} \quad (5)$$

Phase2: We transform $\psi(x) = \xi(x) = S(x)$

Phase3: The Successive Approximation Method is defined by the subsequent formula

$$\psi_{j+1}(x) = f(x) + \int_a^b \mu(x,t)\,\psi_j(t)dt \quad , j \geq 0 \quad (6)$$

The recurrence connection is shown via the use of the progressive approximations approach.

$$\psi_{i,j+1}(x) = f(x) + \sum_{j=1}^{m} \int_a^b \mu_{ij}(x,t)\psi_{i,j}(t)dt \quad i = 1, 2, \dots, n \quad (7)$$

Where the zeroth estimate, $\psi_{i0}(x)$, might be any specific authentically appreciated function. We always start with a beginning guess for $\psi_{i0}(x)$, and for $\psi_{i0}(x)$, we almost always choose either $0,1\ or\ x$ as our starting guess. When this value of $\psi_{i0}(x)$ is inputted into equation (5), some successive estimates of

$\psi_{in}(x), j \geq 1$ will be determined as.

$$\psi_{i1}(x) = f(x) + \sum_{j=1}^{m} \int_a^b \mu_{ij}(x,t)\psi_{j0}dt$$

$$\vdots$$

$$\psi_{i,n+1}(x) = f(x) + \sum_{j=1}^{m} \int_a^b \mu_{ij}(x,t)\psi_{j,n}dt$$

Consequently, the configuration $\psi_i(x), i = 1, \dots, m$ may be obtained by employing

$$\psi_i(x) = \lim_{n\to\infty} \psi_{i,n+1}(x)$$

Phase4: By using of the progressive approximations approach.

$$\psi_{i,j+1}(x) = f(x) + \sum_{j=1}^{m} \int_a^b \mu_{ij}(x,t)\psi_{i,j}(t)dt \quad i = 1, 2, \dots, n \quad (8)$$

Phase5: In the equal, using the a The Successive Approximation Method to the resultant system produces an approximate solution to the problem. (7) as $S_i(t)$, given by equation (5)

Quantitative Illustrations

This section presents three examples to demonstrate the effectiveness and accuracy of the recommended technique. The computed inaccuracies e_i are delineated by $e_i = |\psi_i - S_i|$ where u_i is the precise solution of system (3)and $\psi(t), \xi(t)$ represents an estimated solution to the identical equation. We additionally compute the Least Squares Error. (LSE), defined by the formula.$\sum_{i=0}^{n}(\psi_i - S_i)^2$,All computations are executed with the Matlab program

Example1: Use the successive approximation method to solve the following system of Fredholm integro-differential equations

$$\begin{Bmatrix} \psi'(x) = \frac{7}{15} + 2x + \int_0^1 [t^2\,\psi(t) + t\rho(t)]dt \\ \xi'(x) = \frac{7}{12} - 2x + \int_0^1 [t^3\,\psi(t) + t^2\xi(t)]dt \end{Bmatrix} \quad (9)$$

Integrating both sides the system (9) once from 0 to x and the application of initial circumstances and the subsequent actions as the

$$\begin{Bmatrix} \psi(x) = \frac{7}{15}x + x^2 + \int_0^1 xt^2\,\psi(t)dt + \int_0^1 xt\,\xi(t)dt \\ \xi(x) = \frac{7}{12}x - x^2 + \int_0^1 xt^3\,\psi(t)dt + \int_0^1 x\,t^2\xi(t)dt \end{Bmatrix} \quad (10)$$

The exact solution to this system is provided by $\psi(x) = x + x^2$

Table(1) show a comparison between the exact and numerical solution using a (SAM) for $\psi(x)$ for example (1), dependent on the least square error with $h = 0.2$

Table(1): The Numerical Results for Example (1) for $n = 5$

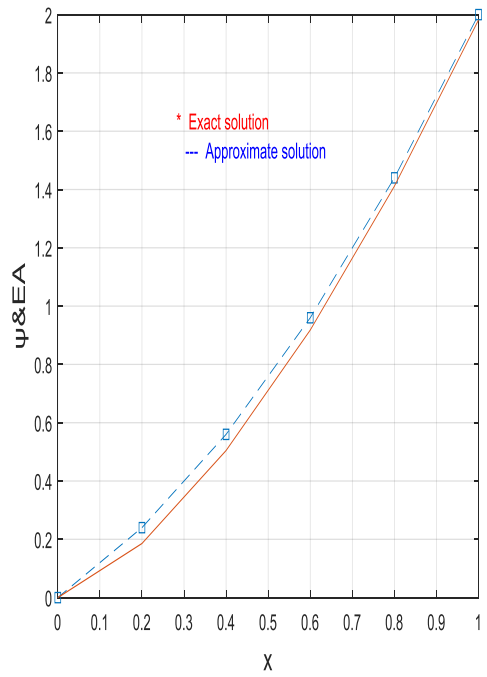
x_i	ψ_i	S_i	$\mid \psi_i - S_i \mid^2$
0.0	0.000000000000	0.0000000000000	0.000000000000
0.2	0.240000000000	0.1856000000000	0.002959360000
0.4	0.560000000000	0.5047000000000	0.003058090000
0.6	0.960000000000	0.9178000000000	0.001780840000
0.8	1.440000000000	1.411400000000	0.000817960000
1.0	2.000000000000	1.981800000000	0.000331240000
LSE			$0.894749000\times 10^{-2}$

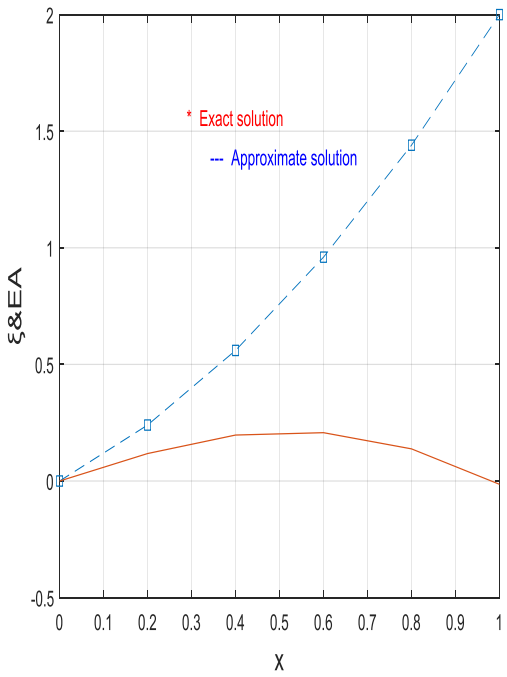
The exact solution to this system is provided by $\xi(x) = x - x^2$

Table(2) show a comparison between the exact and numerical solution using a (SAM) for $\xi(x)$ for example (#), dependent on the least square error with $h = 0.2$

Table(2): The Numerical Results for Example (1) for $n = 5$

x_i	ξ_i	S_i	$\mid \xi_i - S_i \mid^2$
0.0	0.000000000000	0.0000000000000	0.000000000000
0.2	0.160000000000	0.1178000000000	0.001780800000
0.4	0.240000000000	0.1971000000000	0.001840410000
0.6	0.240000000000	0.2073000000000	0.001069290000
0.8	0.160000000000	0.1379000000000	0.000488410000
1.0	0.000000000000	-0.014100000000	0.000198810000
LSE			$0.537772000\times 10^{-2}$





Figure(1): Offers a comparison between the exact answer and the numerical answers obtained using (SAM) interpolation for $\psi(x)$ and $\xi(x)$ in example 1, based on the least square error with $h = 0.2$

Example(2): Use the successive approximation method to solve the following system of Fredholm integro-differential equations

$$\left.\begin{aligned}\psi'(x) &= -0.4597sinx + 0.15853cosx + \int_0^1 [sinx\psi(t) + cosx\,\xi(t)]dt\\ \xi'(x) &= -0.4597cosx + 0.15853sinx + \int_0^1 [cosx\psi(t) + sinx\,\xi(t)]dt\end{aligned}\right\} \quad (11)$$

Integrating both sides the system (11) once or more from 0 to x and the utilizing initial conditions and the subsequent as the

$$\left.\begin{aligned}\psi(x) &= -0.4597 + 0.4597cosx + 0.15853sinx + \int_0^1 [1 - cosx]\psi(t)dt\\ &\quad + \int_0^1 sinx\ \xi(t)dt\\ \xi(x) &= 1.15853 - 0.4597sinx - 0.15853cosx + \int_0^1 sinx\,\psi(t)dt\\ &\quad + \int_0^1 [cosx - 1]\xi(t)dt\end{aligned}\right\} \quad (12$$

When we use the Maclaurin series on $\boldsymbol{sinx}$ and $\boldsymbol{cosx}$, we arrive at the following result:

The exact solution to this system is provided by $\psi(x) = sinx$

Table(3) show a comparison between the exact and numerical solution using a (SAM) for $\psi(\boldsymbol{x})$ for example (2), dependent on the least square error with $h = 0.2$

Table(3): The Numerical Results for Example (2) for $\boldsymbol{n = 5}$

x_i	ψ_i	S_i	$\lvert \psi_i - S_i \rvert^2$
0.0	0.000000000000	0.0000000000000	0.000000000000
0.2	0.003490650000	0.1840000000000	0.032583625000
0.4	0.006981260000	0.4585000000000	0.203869172000
0.6	0.010471784000	0.6586000000000	0.420070184000
0.8	0.013962180000	0.7741000000000	0.577809505000

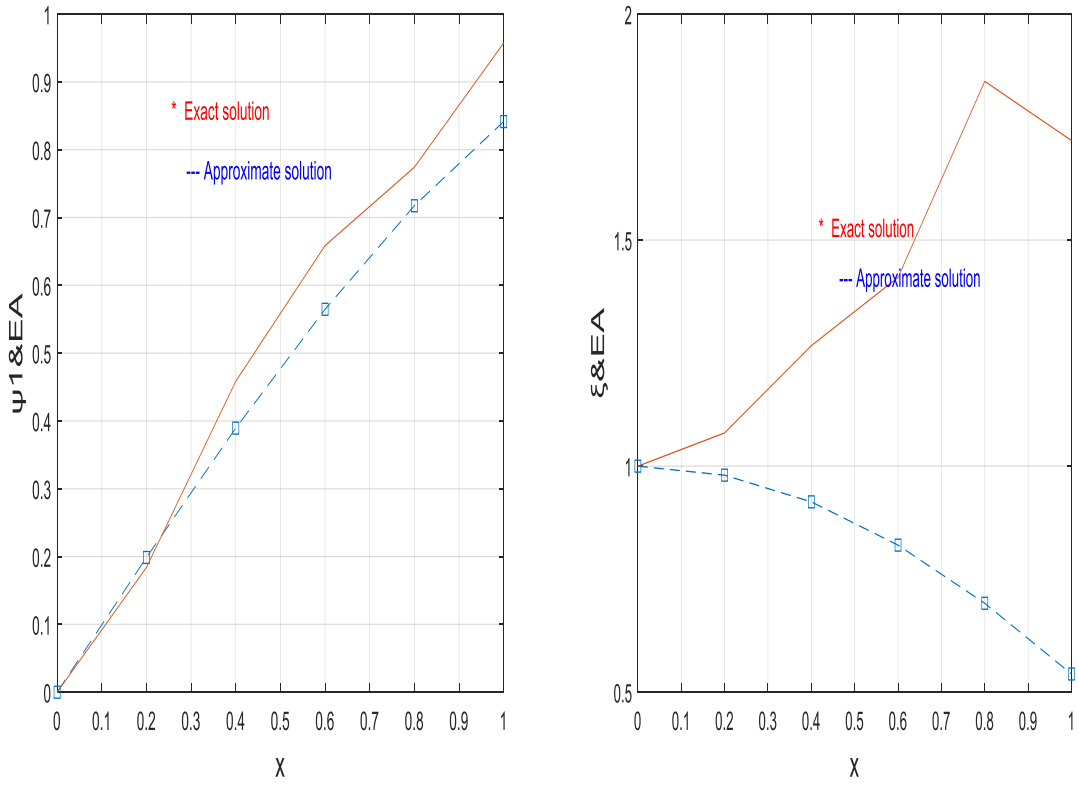
1.0	0.017452406000	0.9573000000000	0.883313499000
LSE			2.117645982000

The exact solution to this system is provided by $\psi(x) = cosx$

Table(4) show a comparison between the exact and numerical solution using a (SAM) for $\xi(x)$ for example (2), dependent on the least square error with $h = 0.2$

Table(4): The Numerical Results for Example (2) for $n = 5$

x_i	ξ_i	S_i	$\lvert\xi_i - S_i\rvert^2$
0.0	1.000000000000	1.0000000000000	0.000000000000
0.2	0.999990000000	1.0735000000000	0.005403720000
0.4	0.999970000000	1.2665000000000	0.071038240000
0.6	0.999940000000	1.4175000000000	0.174356353000
0.8	0.999900000000	1.8512000000000	0.724711690000
1.0	0.999840000000	1.7203000000000	0.519062611000
LSE			1.4946000000000



Figure(2): Offers a comparison between the exact answer and the numerical answers obtained using (SAM) interpolation for $\psi(x)$ and $\xi(x)$ in example 1, based on the least square error with $h = 0.2$

Example(3): Use the successive approximation method to solve the following system of Fredholm integro-differential equations:

$$\left\{\begin{array}{l} \boldsymbol{\psi''(x) = 4e^{-2x} + \frac{e^{-2}}{2} + \frac{e^{-4}}{4} - \frac{3}{4} + \int_0^1 [\psi^2(t) + \xi^2(t)]dt} \\ \boldsymbol{\xi''(x) = e^{-x} + \frac{e^{-2}}{2} - \frac{e^{-4}}{4} - \frac{1}{4} + \int_0^1 [\psi^2(t) - \xi^2(t)]dt} \end{array}\right\} \quad (13)$$

Integrating both sides the system (13) twice from 0 to x and the utilization of initial conditions and the subsequent actions

$$\left.\begin{array}{l} \boldsymbol{\psi(x) = e^{-2x} + (\frac{e^{-2}}{2} + \frac{e^{-4}}{4} - \frac{3}{4})\frac{x^2}{2} + \frac{x^2}{2}\int_0^1 [\psi^2(t) + \xi^2(t)]dt} \\ \boldsymbol{\xi(x) = e^{-x} + (\frac{e^{-2}}{2} - \frac{e^{-4}}{4} - \frac{1}{4})\frac{x^2}{2} + \frac{x^2}{2}\int_0^1 [\psi^2(t) - \xi^2(t)]dt} \end{array}\right\} \quad (14)$$

When we use the Maclaurin series on $\boldsymbol{e^{-x}}$ and $\boldsymbol{e^{-2x}}$, we arrive at the following result

The exact solution to this system is provided by $\psi(x) = e^{-x}$

Table(5) show a comparison between the exact and numerical solution using a (SAM) for $\psi(x)$ for example (3), dependent on the least square error with $h = 0.2$

Table(5): The Numerical Results for Example (3) for $n = 5$

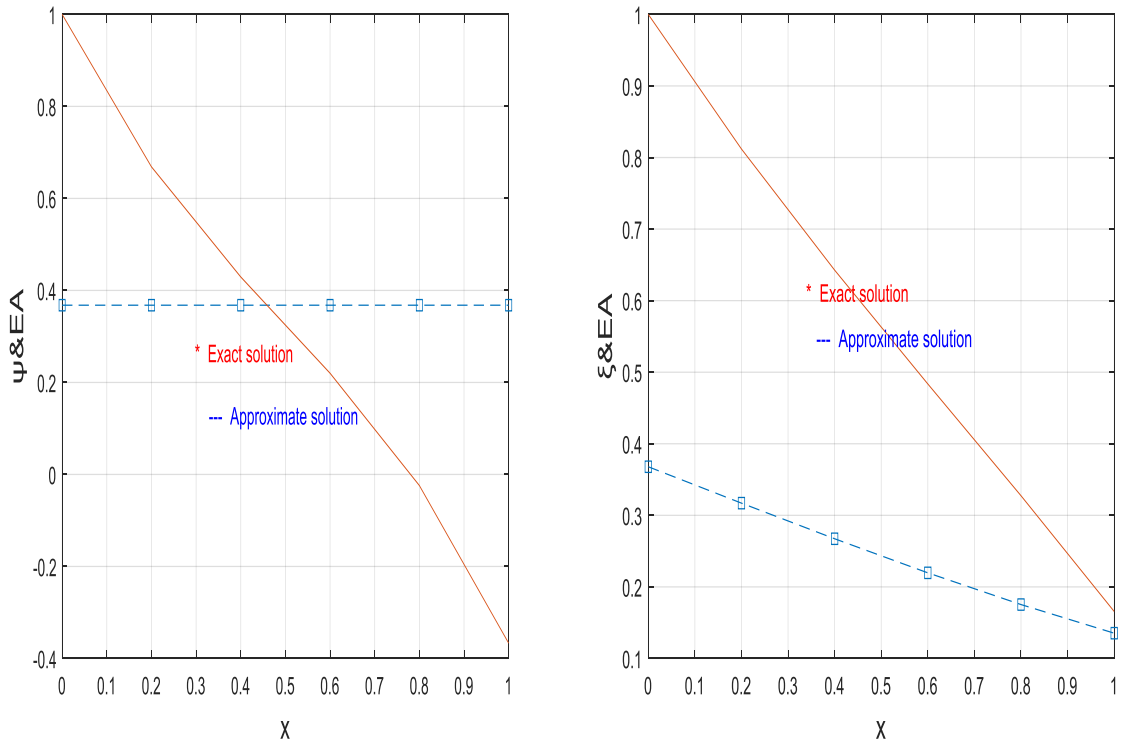
x_i	ψ_i	S_i	$\lvert\psi_i - S_i\rvert^2$
0.0	1.000000000000	1.0000000000000	0.000000000000
0.2	0.818730800000	0.6687000000000	0.022509240000
0.4	0.670320046000	0.4293000000000	0.058090662000
0.6	0.548811857000	0.2200000000000	0.108117237000
0.8	0.449328964000	-0.024100000000	0.224134984000
1.0	0.367879441000	-0.366800000000	0.539753881000
LSE			0.952606004000

The exact solution to this system is provided by $\psi(x) = e^{-2x}$

Table(6) show a comparison between the exact and numerical solution using a (SAM) for $\xi(x)$ for example (3), dependent on the least square error with $h = 0.2$

Table(6): The Numerical Results for Example (3) for $n = 5$

x_i	ξ_i	S_i	$\lvert\xi_i - S_i\rvert^2$
0.0	1.000000000000	1.0000000000000	0.000000000000
0.2	0.670320022600	0.8119000000000	0.020044890000
0.4	0.449328964000	0.6426000000000	0.037353693000
0.6	0.301194211000	0.4836000000000	0.033271871000
0.8	0.201896518000	0.3274000000000	0.015751123000
1.0	0.135335283000	0.1657000000000	0.000922016000
LSE			1.07343593×10^{-1}



Figure(3): Offers a comparison between the exact answer and the numerical answers obtained using (SAM) interpolation for $\psi(x)$ and $\xi(x)$ in example 3, based on the least square error with $h = 0.2$

Table(7): LSE for different values of n for example (1)-(3)

LSEn	$n = 5$	$n = 5$
	ψ_i	ξ_i
Example 1	$0.894749000\times 10^{-2}$	$0.537772000\times 10^{-2}$
Example 2	2.11764598200	1.494600000000
Example 3	9.5260600410^{-1}	1.07343593×10^{-1}

Conclusions

The successive approximation method transforms the coupled Fredholm integro-differential system featuring first and second derivatives into a convergent iterative scheme each iteration requiring the solution of a linear differential problem with updated source terms derived from previous approximations, The method's convergence and accuracy are ensured under mild assumptions on the kernels and operators and it efficiently leverages existing ordinary differential equation solvers Numerical examples confirm that SAM achieves high-precision results often matching analytical solutions in few steps.

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