Solving the prey -predator model by using the Homotopy analytical method

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ABSTRACT:

This paper discusses the homotopy analytical approach for solving system of nonlinear partially differentials equations for prey-predator problems to obtain semi-analytical solutions that are as close as possible to the exact solution and to compute a maximum absolute error and a mean square error, with the inclusion of some figures that illustrate the solution for the prey-predator system.

Keywords: Homotopy Analysis Method, ordinary differential equations, Prey-predator model, mean square error.

1.Introduction:

One of the biological problems is the prey-predator model[1], which is an important natural phenomenon for studying changes in the numbers of prey and predators[2]. This mathematical model is one of the important models that has occupied an important space in studies and research because it is one of the issues for which accurate solutions can be found, as well as numerical, analytical and semi-analytical solutions. In this research, we will study the homotopy analysis technique [3], which is an effective approach for analyzing[4], solving, and discussing the nonlinear models[5],[6] of the following equations:

$$\frac{\partial u}{\partial t} = \delta_1 \Delta u + ru\left(1 - \frac{u}{w}\right) - pvh(ku)$$
$$\frac{\partial v}{\partial t} = \delta_2 \Delta v + qvh(ku) - sv \qquad (1)$$

With the conditions of initial:

$$u(x,0) = u_0 , \qquad v(x,0) = v_0 \qquad (2)$$

where $u(\vec{x},t)$ and $v(\vec{x},t)$ are the community of prey-predator at time (t), and vector position $\vec{x}$.[Δ] is the Laplacian operator in $d \leq 3$ space dimension, and the parameters δ_1,δ_2,r, w, p, k, q, h and s are positive values. and the use of the homotopic analytical method to analyze a nonlinear system of prey-predator partial differential equations is an ideal method for this type of nonlinear systems[7], which has received great attention from researchers who contributed to its development[8],[9], as this method is considered a semi-analytical method, as it depends on the principle of symmetry, and it is also an expansion approach that is irrespective of all coefficients, regardless of their magnitude, making it a suitable option for solving linear equations. and nonlinear differential equations, as this method can freely choose the appropriate basic functions to easily approximate nonlinear differential equations[10], which makes the solution sequential and convergent. Researchers have used this method to solve the problems of the prey-predator system, which we will study in this research and present in detail.

2.Basic idea:

We introduce some basic concepts.

2.1.Maximum Absolute Error[11]**:**

MAE is known as:

$$\|u_{\text{Exact}} - \mu_{\text{n}}(x)\|_{\infty} = max_{a \le x \le b}\{|u_{\text{Exact}} - \mu_{\text{n}}(x)|\}$$

Where u_{Exact} is the analytical solution and $\mu_{\text{n}}(x)$is the approximate solution

2.2.Mean Square Error[11]:

MSE is known because of total of the squared differences between the actual answer. and the approximation solution for vector (i), where [i = 1, 2] divided by the number of points utilized (λ). Its formula is expressed as follows:

$$MSE = \frac{\sum_{i=1}^{m}(u_{Exact}(x_i) - \mu(x_i))^2}{\lambda}$$

3.Basic notion of HAM[3]**:**

Given that we have the equation that follows:

$$\text{N}[\text{z(t)}] = 0, t \ge 0 \qquad (3)$$

where (N) denotes a nonlinear ingredient, z(t) represents an unidentified function, and (t) signifies a standalone variable, etc. As clarity, we leave out limitations or starting points that can be dealt with in the same way. Lian adds to the typical homotopy approach to get a zero-order deformation, creating a mathematical formulation

$$(1 - \text{q})\mathcal{L}[\psi(\text{t; q}) - z_0(t)] = qhH(t)\text{N}[\psi(\text{t; q})] \qquad (4)$$

where q ∈ [0, 1] is the embedding parameter, h is a non-zero auxiliary parameter, H represents a supporting function, L is a supporting linear operator, $z_0(t)$ is an initial estimate of $z(t)$, and $\psi(t;q)$ is an unknown function, respectively.
By utilizing the approach known as homotopy analysis to tackle the prey-predator problem, we derive results for [q=0] and [q=1]:

$$\psi(\text{t}; 0) = \text{z}_0(\text{t}) \quad , \quad \psi(\text{t}; 1) = \text{z}_1(\text{t}) \qquad (5)$$

in that order. Thus, when q rises from 0 to 1, the function $\psi(t;q)$ evolves from the initial estimate $z_0(t)$ to the solution $z(t)$. By enlarging $\psi(t;q)$ into a Taylor series concerning q, we derive:

$$\psi(t;q)=z_0(t)+\sum_{m=1}^{+\infty} z_m(t)\,q^m \quad (6)$$

Where

$$z_m(t)=\frac{1}{m!}\frac{\partial^m \psi(t;q)}{\partial q^m}\Big|_{q=0} \quad (7)$$

If we choose the supporting linear worker, initial suppose helping parameter h, and extra function effectively; sequence (6) convergence occurs to q = 1, we getting:

$$z(t)=\psi(t;1)=z_0(t)+\sum_{m=1}^{+\infty} z_m(t) \quad (8)$$

A single answer of the original nonlinear equation, as established by Lian, is given by h=-1 & H(t)=1. In turn, equation (4) brings it down to:

$$(1-q)\mathcal{L}[\psi(t;q)-z_0(t)]+q\mathrm{N}[\psi(t;q)]=0$$

This employs the homotopy analytical approach, resulting to a directly answer minus the need for a Taylor series. The meaning (7) argues how the zero-order deform equation can be used to get the control equation. subsequently, by identifying the vector.

$$\overrightarrow{z_n}=\{z_0(t),z_1(t),z_2(t),\dots,z_n(t)\} \quad (9)$$

We produce the $m\text{^}th-order$ deformity computation by doing a zero-order deformity formula (4), for a total on [m] times an about regard to a data factor q and then assigning q=0 & segmenting (m!).

$$\mathcal{L}[z_m(t)-\mathrm{X}_m z_{m-1}(t)]=hH(t)R_m(\vec{z}_{m-1}(t)) \quad (10)$$

Where:

$$R_m(\vec{z}_{m-1}(t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\psi(t;q)]}{\partial q^{m-1}}\Big|_{q=0} \quad (11)$$

And

$$\mathrm{X}_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases},$$

It's necessary to note that $z_m(t)$ over$\{\ m\ =\ 1,2,\ldots\}$ is controlled using a linear formula (10), which has linear limit parts arising at original issues that can be easily fixed using computer software like Maple.

4.Application:

By applying [HAM] the system (1) with initial conditions (2), we assume that the solution of system (1), $u(x,t)$ and $v(x,t)$ can be expressed by following a set of base functions:

$$\{t^m\ |m=0,1,2,3,\dots\} \quad (12)$$

in the following form:

$$u(x,t) = \sum_{m=1}^{+\infty} u_m(x,t)q^m \quad , v(x,t) = \sum_{m=1}^{+\infty} v_m(x,t)q^m \qquad (13)$$

Where $u_m, v_m$are coefficient to be determined, in this technique we pick up to gain a rule of answer phrase i.e., the Solve of (1) should appear just like as Answer (13), and other offers should be skipped for (1) and (13). Thun, we can decide pick the linear operator.

$$\mathcal{L}\psi(x,t;q) = \frac{\partial\psi(x,t;q)}{\partial t} \qquad (14)$$

With

$$\mathcal{L}[c_i] = 0 , i = 1,2, \qquad (15)$$

Where $c_i$is constant.

From (1), we be defined a system nonlinear operator:

$$N_1[\psi(x,t;q)] = \frac{\partial\psi_1(x,t;q)}{\partial t} - \delta_1 \frac{\partial^2\psi_1(x,t;q)}{\partial \mathrm{x}^2} + r\psi_1(t;q)\left[1 - \frac{\psi_1(x,t;q)}{w}\right] - \mathrm{p}\psi_2(x,t;q)$$

$$N_2[\psi(x,t;q)] = \frac{\partial\psi_2(x,t;q)}{\partial t} - \delta_2 \frac{\partial^2\psi_2(x,t;q)}{\partial \mathrm{x}^2} + q\psi_2(x,t;q) + s\psi_2(x,t;q) \qquad (16)$$

Using the aforementioned formula (16), we now formulate the zeroth-order deform equations for our system:

$$(1-q)\mathcal{L}[\psi_1(x,t;q) - u_0(x,t)] = qh_1H_1(t)N_1[\psi_i(\mathrm{x},\mathrm{t};\mathrm{q})]$$

$$(1-q)\mathcal{L}[\psi_2(x,t;q) - v_0(x,t)] = qh_2H_2(t)N_2[\psi_i(\mathrm{x},\mathrm{t};\mathrm{q})] \qquad (17)$$

Clearly; q=0 and q=1,

$$\psi_1(x,t;q) = u_0(x,t), \quad \psi_1(x,t;q) = u_1(x,t)$$

$$\psi_2(x,t;q) = v_0(x,t), \quad \psi_2(x,t;q) = v_1(x,t) \qquad (18)$$

Thence, the higher the embed parameter (q) from (0) to (1), $\psi_1(x,t;q)$ *and* $\psi_2(x,t;q)$ varies from the initial guesses $u_0(x,t)$ *and* $v_0(x,t)$ to the solutions $u_1(x,t)$ *and* $v_1(x,t)$ Respectively. spreading $\psi_i(x,t;q)$ in Taylor series with respect to (q), got at:

$$\psi_1(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_i(x,t)q^m, \quad \psi_2(x,t;q) = v_0(x,t) + \sum_{m=1}^{+\infty} v_i(x,t)q^m \qquad (19)$$

Where

$$\mathrm{u_m}(x,t) = \frac{1}{m!}\frac{\partial^m\psi_1(x,t;q)}{\partial q^m}\Big\lfloor_{q=0} \quad , \quad \mathrm{v_m}(x,t) = \frac{1}{m!}\frac{\partial^m\psi_2(x,t;q)}{\partial q^m}\Big\lfloor_{q=0}$$

Should define the vectors:

$$\overrightarrow{u_n} = \{u_0(x,t), u_1(x,t), u_2(x,t), \dots, u_n(x,t)\} \ , \quad \overrightarrow{v_n} = \{v_0(x,t), v_1(x,t), v_2(x,t), \dots, v_n(x,t)\}$$

Therefor the m^{th}-order deformation equations are:

$$\mathcal{L}[u_m(x,t) - \mathrm{X}_m u_{m-1}(x,t)] = h_1 R_{1,m}(\vec{u}_{m-1})$$
$$\mathcal{L}[v_m(x,t) - \mathrm{X}_m v_{m-1}(x,t)] = h_2 R_{2,m}(\vec{v}_{m-1}) \qquad (20)$$

Where

$$R_{1,m}(\vec{u}_{m-1}) = u'_{m-1}(x,t) - \delta_1 u''_{m-1}(x,t) - r\sum_{i=0}^{m-1} u_{m-1}(x,t)[1 - \frac{u_{m-1}}{w}] + p\sum_{i=0}^{m-1} v_{m-1}(x,t)$$
$$R_{2,m}(\vec{v}_{m-1}) = v'_{m-1}(x,t) - \delta_2 v''_{m-1}(x,t) - q\sum_{j=0}^{m-1} v_{m-1}(x,t) + s\sum_{j=0}^{m-1} v_{m-1}(x,t)$$

Will now, for all $m \geq 1$, getting us the solution of the $m^{th} - order$ deformation equations (20):

$$u_m(x,t) = \mathrm{X}_m u_{m-1}(x,t) + h_1\int_0^t R_{1,m}(\vec{u}_{m-1}(\tau))d\tau + c_1$$
$$v_m(x,t) = \mathrm{X}_m v_{m-1}(x,t) + h_2\int_0^t R_{2,m}(\vec{v}_{m-1}(\tau))d\tau + c_2$$

Where $[c_1] and [c_2]$are terms of integration and its determined by the initial terms(2).

5.Exact solution of [HAM][12]**:**

To find some semi-analytical solutions by [HAM][3] [13] according to the following exact solutions:

$$u_{Exact}(x,t) = \frac{1}{1 + e^{(x+t)}}$$
$$v_{Exact}(x,t) = \frac{1}{e^{(-x-t)}}$$

We replace each t in the previous exact solution with zero to obtain the initial conditions for the system equations as follows:

$$u(x,0) = \frac{1}{1 + e^{(x)}}$$
$$v(x,0) = \frac{1}{e^{(-x)}}$$

Now, let us assume the values of the variables in system (1) as follows:

$$\delta_1 = 1, \delta_2 = 1, w = 1, r = 1, k = 1, p = 1, q = 1, s = 1$$

Now, we can find the second iterations:

$$u_1(x,t) = \int_0^t \left(\frac{\partial}{\partial t} u_0(x,t)\right) - \left(\frac{\partial^2}{\partial x^2} u_0(x,t)\right) - u_0(x,t)\big(1 - u_0(x,t)\big) + v_0(x,t)dt$$

$$u_1(x,t) = \frac{e^x t(e^x + 1 + 3e^{2x} + e^{3x}}{(1+e^x)^3},$$

$$v_1(x,t) = \int_0^t \left(\frac{\partial}{\partial t} v_0(x,t)\right) - \left(\frac{\partial^2}{\partial x^2} v_0(x,t)\right) - v_0(x,t) + v_0(x,t)dt$$

$$v_1(x,t) = -e^x t\,,$$

And by using the following formula:

$$u_{i+1}(x,t) = u_i(x,t) + \int_0^t \left[\left(\frac{\partial}{\partial t} u_i(x,t)\right) - \left(\frac{\partial^2}{\partial x^2} u_i(x,t)\right) - u_i(x,t)\big(1 - u_i(x,t)\big) + v_i(x,t)\right] dt$$

$$v_{i+1}(x,t) = v_i(x,t) + \int_0^t \left(\frac{\partial}{\partial t} v_i(x,t)\right) - \left(\frac{\partial^2}{\partial x^2} v_i(x,t)\right) - v_i(x,t) + v_i(x,t)d$$

We can find third iteration and all remaining iterations:

$$u_2(x,t) = \frac{\begin{matrix}\frac{1}{6} e^x t(48e^x + 108e^{2x} + 168e^{3x} + 156e^{4x} + 12 + 72e^{5x} + 12e^{6x} \\ +14e^{3x}t^2 + 4e^{2x}t^2 + 16e^{4x}t^2 + 22e^{5x}t^2 + 2e^x t^2 + 12e^{6x}t^2 \\ +2e^{7x}t^2 - 9t - 24e^x t - 135e^{2x}t - 198te^{3x} - 123te^{4x} - 54te^{5x} - 9te^{6x})\end{matrix}}{(1+e^x)^6}$$

$$v_2(x,t) = -2e^x t + \frac{1}{2} e^x t^2$$

And fourth iteration is:

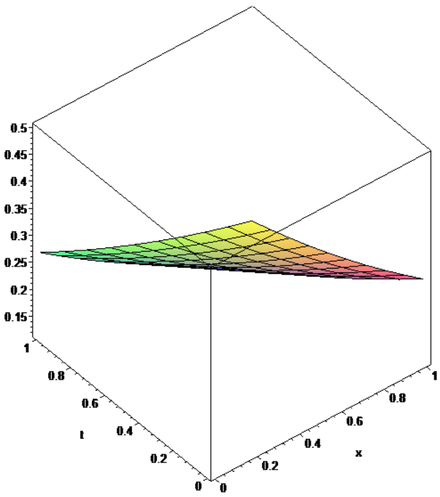
$$u_3(x,t) = \frac{1}{1260} e^x t(5040 + 50400e^x - 7560t - 65520e^x t + 241920e^{2x} + 745920e^{3x} + 1470t^2 + 9660e^x t^2 + 499800e^{3x}t^2 + 117180e^{2x}2t^2 + 1085490e^{4x}t^2 + 1617840e^{5x}t^2 + 2058840e^{6x}t^2 + 2273040e^{7x}7t^2 - 347760e^{2x}t - 1300320te^{3x} - 3318840te^{4x} - 5775840te^{5x} - 6985440te^{6x} + 1648080e^{4x} + 2721600e^{5x} + 3144960e^{7x}7 + 3386880e^{6x} - 5987520e^{7x}t + 567e^x t^4 - 2415e^x t^3 + 1952370e^{8x}t^2 + 1229340e^{9x}t^2 + 559020e^{10x}t^2 + 173880e^{11x}t^2 + 31710e^{12x}t^2 + 2520e^{13x}t^2 - 3681720e^{8x}t - 1617840te^{9x} - 488880te^{10x}10 - 90720te^{11x} - 7560te^{12x} + 80e^{4x}4t^6 + 360e^{5x}5t^6 + 880e^{6x}t^6 + 2060e^{7x}t^6 + 3360e^{8x}8t^6 + 4880e^{9x}t^6 + 5280e^{10x}t^6 + 4620e^{11x}t^6 + 2960e^{12x}t^6 + 1160e^{13x}t^6 + 240e^{14x}t^6 + 20e^{15x}t^6 - 980e^{3x}t^5 - 5740e^{4x}t^5 - 16520e^{5x}t^5 - 40950e^{6x}t^5 - 71960e^{7x}t^5 - 98000e^{8x}t^5 - 102480e^{9x}9t^5 - 73990e^{10x}t^5 - 37380e^{11x}t^5 - 12740e^{12x}t^5 - 2520e^{13x}13t^5 - 210e^{14x}t^5 + 3360e^{2x}t^4 + 23058e^{3x}t^4 + 2131920e^{8x} + 1018080e^{9x} + 322560e^{10x} + 60480e^{11x} + 5040e^{12x} + 78372e^{4x}t^4 + 232449e^{5x}t^4 + 471744e^{6x}t^4 + 610848e^{7x}t^4 + 560280e^{8x}t^4 + 393561e^{9x}t^4 + 215712e^{10x}t^4 + 91686e^{11x}t^4 + 26964e^{12x}t^4 + 4599e^{13x}t^4 + 336e^{14x}t^4 - 78330e^{3x}t^3 - 276780e^{4x}t^3 - 672525e^{5x}t^3 - 1223880e^{6x}t^3 - 1645560e^{7x}t^3 - 1567440e^{8x}t^3 - 1041285e^{9x}t^3 - 483420e^{10x}t^3 - 152670e^{11x}t^3 - 28980e^{12x}t^3 - 2415e^{13x}t^3 + 20e^{3x}t^6 - 210e^{2x}t^5 - 14700e^{2x}t^3)/(1 + e^x)^{12}$$

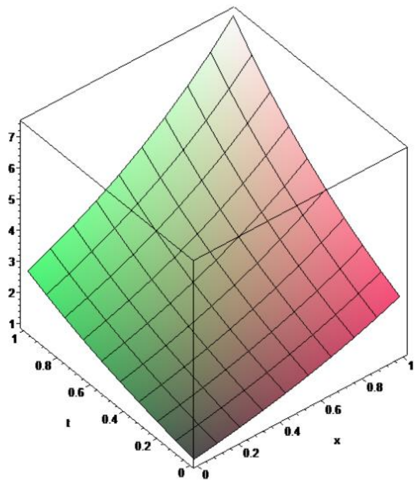
$$v_3(x,t) = -4e^x t + 2e^x t^2 - \frac{1}{6}e^x t^3$$

Table1: result MAE (The first column represents the difference between the exact solution and the approximate solution of the equation $u(x,t)$. The second column represents the difference between the exact solution and the approximate solution of the equation $v(x,t)$, when t=0.001).

x	$(Exact - HAM)_{u(x,t)}$	$(Exact - HAM)_{v(x,t)}$
0.0	0.011475	0.015992
0.1	0.012872	0.017673
0.2	0.014455	0.019532
0.3	0.016245	0.021586

0.4	0.018265	0.023857
0.5	0.020536	0.026366
0.6	0.023083	0.029139
0.7	0.025933	0.032203
0.8	0.029113	0.035590
0.9	0.032654	0.039333
1.0	0.036590	0.043470
MSE	0.005983	0.009269





1 **2**

Figures (1) and (2) show the change in prey and predator community size for three iterations, when $u_3(x,t)$ and $v_3(x,t)$.

6.Conclusions:

This discussion focuses on the homotopy method and its application in deriving findings from a set of nonlinear partial equations within the prey-predator model. Specifically, it addresses the computation of absolute error values for both u(x,t) & v(x,t), as well as an estimation of a mean square error for both functions.

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