New Results on Fourth-Hankel Determinant of a Certain Subclass of Analytic Functions

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New Results on Fourth-Hankel Determinant of a Certain Subclass of Analytic Functions

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Abstract

This paper studies the fourth Hankel determinaf for a subclass of analytic functions, denoted by $\mathrm{N}(\alpha, \mu, e)$, defined via athir-order differential subordination involving an exponential function. Sharp coefficient bounds for lanl, where $n = 2, \ldots, 7$, are obtained, and an upper bound for the fourth Hankel determinant $H_4(1)$ is established, contributing to the theary of geometric function classes.

Keywords: Superordination, Hankel determinant, A Schwarz function, Chebyshev polynomials, Analytic function.

1. Introduction

Let p denote the class of analytic functions p , normalize by

$$\wp(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \qquad (1)$$

and satisfying the condition $Re\{\wp(z)\} > 0$ in D

It is well known (see [2,3,4, 19,24]) that if $p(z) \in \mathcal{P}$, then there exists a Schwarz function $w(z)$, analytic in such that $w(0) = 0$ *and* $|w(z)| < 1$, with

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \quad (z \in D).$$

Recently, Mendiratta et al. [15] introduced the following subclass of analytic functions associated with exponential function. Earlions

Ma and Minda [14] defined subclass of starlike and convex functions using t principle of subordination. Inparticnlar, they introduce for the

$$S^*(\Phi) = \left\{ f \in A \colon \frac{zf'(z)}{f(z)} \prec \Phi(\mathrm{z}), \;\; z \in U \right\}$$

and

$$G^*(\Phi) = \left\{ f \in A \colon 1 + \frac{zf''(z)}{f(z)} \prec \Phi(\mathrm{z}), \;\; z \in U \right\}.$$

Mendiratta et al. [15] father introduced the class ors

$$S_e^* = \left\{ f \in A \colon \frac{zf'(z)}{f(z)} \prec e^z, \;\; z \in U \right\}. \tag{2}$$

While the associated class

$$G_e^* = \left\{ f \in A \colon 1 + \frac{zf''(z)}{f(z)} \prec e^z, \quad z \in U \right\}. \tag{3}$$

Was introduced by using an Alexander type relation [13]

These classes are known to de symmetric with respect do the real axis

$$\mathrm{H}_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} \cdots & a_{n+2q-2} \end{vmatrix}, (a_1 = 1). \tag{4}$$

The Hankel determinant plays a significant role in the theory of singularities [10] in the analysis of power series with integer coefficients [7,17] several estimates of $\mathrm{H}_q(n)$ have been obtained for various sub classes of univalent and bi-univalen function. The case $\mathrm{H}_2(1) = a_3 - a_2^2$ is the classic Fekete–szego functional while $\mathrm{H}_2(2) = a_2a_4 - a_3^2$ has been studied for bi– starlike and bi – convex functions ([3,45,62]). Krisha [13] provided asharp esti for the Bazileric class. More recently, Srivastavaetal. [21],obtained bounds for $\mathrm{H}_2(2)$ in the class of bi-univalent functions involving symmetric q-derivative[23], the authors investigated Hankel and Toeplitz determinants in subfamilies of q-starlike functions associated with conic domain(see also [6,16,22]).

For functions of form (1), the third Hankel determinant is given as:

$$H_3(1) = -a_5a_2^2 + 2a_2a_3a_4 - a_3^3 + a_3a_5 - a_4^2.$$

Interesting results on $H_3(1)$ were established by Babalola [5], motivating further investigations.

The fourth Hankel determinant has also been studied in for a certain subclass of starlike and convex functions [20]**.** In this direction, we introduced a new subclasses of analytic functions using third-order differential subordination involving the exponential function and derive sharp bounds for the fourth Hankel determinant $H_4(1)$ for functions in this class.

2-Preliminaries

The definitions and lemmas presented below will be used to establish the main results work .

Definition2.1: [20] Let f be a function f of the form (1) .The fourth Hankel determinant of f is defined as:

$$H_4(1) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix} = -a_4t_1 + a_5t_2 - a_6t_3 + a_7t_4, \qquad (5)$$

Where the coefficients ti are defined by:

$$|t_1| = |a_2||a_4a_6 - a_5^2| + |a_3||a_3a_6 - a_4a_5| - |a_4||a_3a_5 - a_4^2|,$$

$$|t_2| = |a_4a_6 - a_5^2| - |a_2||a_3a_6 - a_4a_5| + |a_3||a_3a_5 - a_4^2|, \qquad (6)$$

$$|t_3| = |a_3a_6 - a_4a_5| + |a_2||a_2a_6 - a_3a_5| - |a_4||a_2a_4 - a_3^2|,$$

$|t_4| = |a_3||a_2a_4 - a_2^2| - |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$

The Chebyshev polynomials of the first and second kind are defined on the interral $(-1,1)$ for a real variable x an foiiows:

$$T_n(x) = \cos(n \arccos x)$$

and

$$U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)} = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}},$$

Now, consider the function

$$H(t,z) = \frac{1}{1-2t+z^2}, t \in \left(\frac{1}{2},1\right), z \in \mathbb{U}.$$

It is well – known that if $t = \cos\alpha$, $\alpha \in \left(0, \frac{\pi}{3}\right)$, then

$$H(t,z) = 1 + \sum_{n=1}^{\infty} \frac{\sin[(n+1)\alpha]}{\sin\alpha} z^n$$

$$= 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + (8\cos^3\alpha - 4\cos\alpha)z^3 + \cdots, z \in U,$$

that is

$$H(t,z) = 1 + U_1(t)\,z + U_2(t)z^2 + U_3(t)z^3 + U_4(t)z^4 + \cdots, t \in \left(\frac{1}{2},1\right), z \in U$$

where

$$U_n(t) = \frac{\sin[(n+1)\arccos t]}{\sqrt{1-t^2}}, n \in \mathbb{N},$$

are the Chebyshev polynomials of the second kind . There polynomials satisfy the recurrence relation:

$$U_{n+1}(t) = 2tU_n(t) - U_{n-2}(t).$$

specifically

$$U_1(t) = 2t, U_2(t) = 4t^2 - 1,, U_3(t) = 8t^3 - 4t, (\text{for each } n \in \mathbb{N}).$$

Mendiratta et al. [79] discussed the subclass S_1^* of analytic functions associated with exponential function.

Lemma2. 1: [15] If function $f \in S_1^*$ is of the form (2), then

$$|a_2| \leq 1, |a_3| \leq \frac{3}{4}, |a_4| \leq \frac{17}{36}, |a_5| \leq 1,$$

where S_1^* denote the class of analytic functions to third Hankel determinant.

Lemma2. 2: [20] If the function $f \in S_e^*$ and of the form (1), then

$$|a_2| \leq 1, |a_3| \leq \frac{3}{4}, |a_4| \leq \frac{1}{18}, |a_5| \leq \frac{1}{96}, |a_6| \leq \frac{1}{600}, |a_7| \leq \frac{2401}{3600}.$$

Lemma2. 3: [20] If the function $f \in G_e^*$ and of the form (1), then

$$|a_2| \leq \frac{1}{2}, |a_3| \leq \frac{3}{12}, |a_4| \leq \frac{1}{72}, |a_5| \leq \frac{1}{480}, |a_6| \leq \frac{1}{3\,600}, |a_7| \leq \frac{343}{3600}.$$

Here, we discuss a new subclass of analytic functions using the subordination.

Lemma 2.4: [11] If P be a class of all analytic functions $p(\mathrm{z})$ of the form:

$$p(\mathrm{z}) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{7}$$

with $p(0)$=1 and $\Re e\{\, p(\mathrm{z})\} > 0$ for all $\mathrm{z} \in U$. Then $|p_n| \leq 2$, for every $(n = 1,2,3, \dots)$.This disparity is sharp for each n

Definition 2.2: A function $f \in A$ given by (1) is said to be in the class $\mathcal{N}(\alpha, \mu, e)$ if the following condition holds:

$$\frac{1}{\alpha}\left[\frac{f(z)}{z} + (1-\mu)f'(z) + z\mu f''(z) + z^2 f'''(z)\right] \prec \mathcal{L}(e, z), \tag{8}$$

where $\mu \geq 0$,$\alpha \in \mathbb{C} \setminus \{0\}$ and

$$\mathcal{L}(e, z) = 1 + U_1(t)\, z + U_2(t) z^2 + U_3(t) z^3 + U_4(t) z^4 + \cdots ,$$

$$e \in \left(\frac{1}{2}, 1\right), z \in U.$$

The main findings of our current inquiry will now be stated and proven.

We begin here by finding the estimates on the coefficients$|a_n|$ and $n = 2,3,4,5,6,7$, for functions in the class $\mathcal{N}(\alpha, \mu, e)$.

Theorem 1: Let f be a function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \qquad (z \in U)$$

And suppose that f belongs to class $\mathcal{N}(\alpha,\mu,e)$

A Then the follow sharp coefficient bounds hold:

$$|a_2| \le \frac{4}{3-2\mu}\alpha\,, |a_3| \le \frac{12}{7}\alpha\,, |a_4| \le \frac{36}{13+12\mu}\alpha, \qquad |a_5| \le \frac{108}{21+40\mu}\alpha$$

$$|a_6| \le \frac{216}{31+90\mu}\alpha\,, |a_7| \le \frac{1044}{43+168\mu}\alpha,$$

where $\mu \ge 0$ and $\alpha \in \mathbb{C}\setminus\{0\}$.

Proof: A ssum that $f \in \mathcal{N}(\alpha,\mu,e)$, then there exists an analytic function E defined in the following subordination relation holds:

$$\frac{1}{\alpha}\left[\frac{f(z)}{z} + z(1-\mu)f''(z) + z^2\mu f'''(z)\right] = \mathcal{L}(e,\mathrm{E}(z)).$$

Where $L(e,z) = 1 + u_1(t)z + u_2(t)z^2 + \cdots \ldots..$ is generating function involving chebysher polnomiais of the second by the principle of subordination, there exists a Schwarz function $\mathrm{E}(z)$, given by

$$\mathrm{E}(z) = \sum_{n=1}^{\infty} p_n\, z^n\,,\ \ z \in U,$$

Such that the function $\mathrm{B}(z) = \frac{1+\mathrm{E}(z)}{1-\mathrm{E}(z)}$

Has the power series expansion

$$\begin{aligned}\mathrm{B}(z) = 1 &+ 2\,p_1 z + 2(\,p_2 + p_1^2)z^2 + 2[\,p_3 + p_1(2\,p_2 + p_1^2)]z^3\\
&+ 2[\,p_4 + p_2^2 + p_1^2(3\,p_2 + p_1^2) + 2\,p_1\,p_3]z^4\\
&+ 2\big[\,p_5 + 2\,p_2(\,p_3 + 2p_1^3) + 3p_1(p_1p_3 + p_2^2)\\
&+ \big(p_1(2p_4 + p_1^4)\big)\big]z^5\\
&+ 2[\,p_6 + p_1^3(4\,p_3 + p_1^3) + p_1^2(3p_4 + 5p_2p_1^2) + 2p_1(p_5 + 3p_2p_3)\\
&+ p_3^2 + p_2^2(\,p_2 + 6p_1^2) + 2\,p_2\,p_4]z^6\\
&+ \cdots. \qquad (9)\end{aligned}$$

Now ,Since f is the stranded class A, We Can expand the left-hand side of operator as follows:

$$\frac{1}{\alpha}\left[\frac{f(z)}{z}+z(1-\mu)f''(z)+z^2\mu f'''(z)\right]$$
$$=\frac{1}{\alpha}[1+(3-2\mu)a_2z+7a_3z^2+(13+12\mu)a_4z^3$$
$$+(21+40\mu)a_5z^4+(31+90\mu)a_6z^5+(43+168\mu)a_7z^6$$
$$+\cdots]. \qquad (10)$$

comparing the coefficients of z^n form the two expansion $(9) and (10)$, we obtain

$$\frac{1}{\alpha}\big((3-2\mu)a_2\big)=2\,p_1,$$
$$\frac{1}{\alpha}(7a_3)=2(\,p_2+p_1^2),$$
$$\frac{1}{\alpha}\big((13+12\mu)a_4\big)=2[\,p_3+p_1(2\,p_2+p_1^2)],$$
$$\frac{1}{\alpha}\big((21+40\mu)a_5\big)=2[\,p_4+p_2^2+p_1^2(3\,p_2+p_1^2)+2\,p_1\,p_3],$$
$$\frac{1}{\alpha}\big((31+90\mu)a_6\big)=2\big[\,p_5+2\,p_2(\,p_3+2p_1^3)+3p_1(p_1p_3+p_2^2)+\big(p_1(2p_4+p_1^4)\big)\big],$$
$$\frac{1}{\alpha}\big((43+168\mu)a_7\big)$$
$$=2[\,p_6+p_1^3(4\,p_3+p_1^3)+p_1^2(3p_4+5p_2p_1^2)+2p_1(p_5+3p_2p_3)$$
$$+p_3^2+p_2^2(\,p_2+6p_1^2)+2\,p_2\,p_4].$$

Now, by applying the standard bound $|p_n|\le 1$ for Schwarz function and applying Lemma 4, we obtain

$$|a_2|\le\frac{4}{(3-2\mu)}\alpha\,, \qquad (11)$$
$$|a_3|\le\frac{12}{7}\alpha\,, \qquad (12)$$
$$|a_4|\le\frac{36}{13+12\mu}\alpha, \qquad (13)$$
$$|a_5|\le\frac{108}{21+40\mu}\alpha, \qquad (14)$$

$$|a_6| \le \frac{216}{31+90\mu}\alpha, \qquad (15)$$

$$|a_7| \le \frac{1044}{43+168\mu}\alpha. \qquad (16)$$

This complete the proof . □

In the following theorem, estimates on $|H_4(1)|$ are determined for $f \in \mathcal{N}(\alpha,\mu,e)$.

Theorem 2: Let f be a function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \qquad (z \in U)$$

And suppose that f belongs to class $\mathcal{N}(\alpha,\mu,e)$. then the following estivate for the fourth Hankel determinant $H_4(1)$ holds:

$$|H_4(1)|$$

$$\le \frac{-111974(2\mu-3)(90\mu+31)(40\mu+21)\alpha^4 L_1(\delta,u) + 139968(90\mu+31)(12\mu+13)^{2}}{L(\delta,u)}$$

$$- \frac{93312(12\mu+13)^2(40\mu+21)^2\alpha^3 L_3(\delta,u) + 672(90\mu+31)^2\alpha^2 L_4(\delta,u)}{L(\delta,u)},$$

where

$$L(\delta,u) = 49(12\mu+13)^4(2\mu-3)^2(90\mu+31)(40\mu+21)^3$$

$$L_1(\delta,u) = 5529600\mu^6 - 6289920\mu^5 - 7939584\mu^4 + 3821168\mu^3 + 19330358\mu^2 + 13831923\mu + 2364054,$$

$$L_2(\delta,u) = 12441600\alpha\mu^5 + (7822080\alpha - 141120)\mu^4 + (2282496\alpha - 4407648)\mu^3 + (-9231232\alpha + 936684)\mu^2 + (-7799688\alpha + 7738668)\mu - 1318212\alpha + 1874691,$$

$$L_3(\delta,u) = (2073600\alpha + 967680)\mu^5 + (5760\alpha - 2203488)\mu^4 + (3538848\alpha - 405048)\mu^3 + (1919952\alpha + 2910348)\mu^2 + (-908008\alpha - 744282)\mu - 163380\alpha - 257985,$$

$$L_4(\delta,u) = -7776\mu^4 + (46080\alpha + 21600)\mu^3 + (52896\alpha - 13500)\mu^2 + (2216\alpha - 324)\mu - 1092\alpha - 2673.$$

Proof: Let f be a function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \qquad (z \in U)$$

And suppose that f belongs to class $\mathcal{N}(\alpha, \mu, e)$

By Deficient (2,1) the fourth Hankel determinant $H_4(1)$ is given by

$$|H_4(1)| = |a_7 t_4 - a_6 t_3 + a_5 t_2 - a_4 t_1|,$$

Where the coefficients t_1, t_2, t_3, t_4 are defined by

$|t_1| = |a_2||a_4 a_6 - a_5^2| + |a_3||a_3 a_6 - a_4 a_5| - |a_4||a_3 a_5 - a_4^2|,$

$|t_2| = |a_4 a_6 - a_5^2| - |a_2||a_3 a_6 - a_4 a_5| + |a_3||a_3 a_5 - a_4^2|,$

$|t_3| = |a_3 a_6 - a_4 a_5| + |a_2||a_2 a_6 - a_3 a_5| - |a_4||a_2 a_4 - a_3^2|,$

$|t_4| = |a_3||a_2 a_4 - a_2^2| - |a_4||a_4 - a_2 a_3| + |a_5||a_3 - a_2^2|.$

Inserting $(11) - (16)$ in (7), we get

$$|t_1| = \frac{31104\alpha^3 L_1(\delta,u)}{R_1(\delta,u)}, \qquad (17)$$

$$|t_2| = \frac{1296\alpha^2 L_2(\delta,u)}{R_2(\delta,u)}, \qquad (18)$$

$$|t_3| = \frac{432\alpha^2 L_3(\delta,u)}{R_3(\delta,u)}, \qquad (19)$$

$$|t_4| = -\frac{96\alpha^2 L_4(\delta,u)}{R_4(\delta,u)}, \qquad (20)$$

where

$$R_1(\delta, u) = 43(2\mu - 3)(90\mu + 31)(40\mu + 21)^2(12\mu + 13)^3,$$

$$R_2(\delta, u) = 49(90\mu + 31)(2\mu - 3)(12\mu + 13)^2(40\mu + 21)^2,$$

$$R_3(\delta, u) = 49(90\mu + 31)(40\mu + 21)(2\mu - 3)^2(12\mu + 13)^2,$$

$$R_4(\delta, u) = 7(40\mu + 21)(2\mu - 3)^2(12\mu + 13)^2.$$

Using (17) – (20) in (5), we get

$$|H_4(1)| \le \frac{-111974(2\mu - 3)(90\mu + 31)(40\mu + 21)\alpha^4 L_1(\delta, u) + 139968(90\mu + 31)(12\mu + 13)^2\alpha^3 L_2(\delta, u)}{L(\delta, u)}$$

$$- \frac{93312(12\mu + 13)^2(40\mu + 21)^2\alpha^3 L_3(\delta, u) + 672(90\mu + 31)^2\alpha^2 L_4(\delta, u)}{L(\delta, u)},$$

where

$$L_1(\delta,u) = 5529600\mu^6 - 6289920\mu^5 - 7939584\mu^4 + 3821168\mu^3 + 19330358\mu^2 + 13831923\mu + 2364054,$$

$$L_2(\delta,u) = 12441600\alpha\mu^5 + (7822080\alpha - 141120)\mu^4 + (2282496\alpha - 4407648)\mu^3 + (-9231232\alpha + 936684)\mu^2 + (-7799688\alpha + 7738668)\mu - 1318212\alpha + 1874691,$$

$$L_3(\delta,u) = (2073600\alpha + 967680)\mu^5 + (5760\alpha - 2203488)\mu^4 + (3538848\alpha - 405048)\mu^3 + (1919952\alpha + 2910348)\mu^2 + (-908008\alpha - 744282)\mu - 163380\alpha - 257985,$$

$L_4(\delta,u) = -7776\mu^4 + (46080\alpha + 21600)\mu^3 + (52896\alpha - 13500)\mu^2 + (2216\alpha - 324)\mu - 1092\alpha - 2673.$ □

In case $\mu = 0$, we get the following Corollary.

Corollary 1: If function f of form (1) belongs to the subclass $\mathcal{N}(\alpha,\mu,e)$, then

$$|H_4(1)| \leq\approx -\frac{825676789240}{911067339}\alpha^4 + \frac{3869374792502}{4415172489}\alpha^3 + \frac{259842}{480024727}\alpha^2.$$

In case $\alpha = 1\ and\ \mu = 0$, we get the following Corollary.

Corollary 2: If function f of form (1) belongs to the subclass $\mathcal{N}(\alpha,\mu,e)$, then

$$|H_4(1)| \leq\approx -29.89227843\,.$$

By applying Lemma 1 and using $|a_6|, |a_7|$ from Theorem 2 in fourth Hankel determinant (7), the following theorem emerges:

Theorem 3: let the function $f \in \mathcal{N}(\alpha,\mu,e)$ be of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n\ , \qquad (z \in U)$$

Then , the fourth Hankel determinant satisfies the inequality

$$|H_4(1)| \leq -\Upsilon_1(\delta,u) + \Upsilon_2(\delta,u) - \Upsilon_3(\delta,u) + \Upsilon_4(\delta,u),$$

Where the farms are defined by:

$$\Upsilon_1(\delta,u)=\frac{2533\alpha}{24(31+90u)}-\frac{1271447}{1679616},$$

$$\Upsilon_2(\delta,u)=-60*\frac{-60\alpha}{31+90u}-\frac{229}{1728},$$

$$\Upsilon_3(\delta,u)=81648*\frac{81648\alpha^2}{(31+90u)^2}-\frac{6115\alpha}{24(31+90u)},$$

$$\Upsilon_4(\delta,u)=\frac{-19343\alpha}{36(43+168u)}$$

Proof: Let $f\in\mathcal{N}(\alpha,\mu,e)$. Then the fourth Hankel determinant can be rewrite as:

$|H_4(1)|=-a_4t_1+a_5t_2-a_6t_3+a_7t_4$,

where

$|t_1|=|a_2||a_4a_6-a_5^2|+|a_3||a_3a_6-a_4a_5|-|a_4||a_3a_5-a_4^2|,$

$|t_2|=|a_4a_6-a_5^2|-|a_2||a_3a_6-a_4a_5|+|a_3||a_3a_5-a_4^2|,$

$|t_3|=|a_3a_6-a_4a_5|+|a_2||a_2a_6-a_3a_5|-|a_4||a_2a_4-a_3^2|,$

$|t_4|=|a_3||a_2a_4-a_2^2|-|a_4||a_4-a_2a_3|+|a_5||a_3-a_2^2|.$

By applying Lemma 1 and using $|a_6|,|a_7|$ from Theorem 2 in fourth Hankel determinant (7), we get

$|t_1|=|a_2||a_4a_6-a_5^2|+|a_3||a_3a_6-a_4a_5|-|a_4||a_3a_5-a_4^2|,$

$$|t_1|=\frac{447\alpha}{2(31+90u)}-\frac{74791}{46656}, \qquad (21)$$

$|t_2|=|a_4a_6-a_5^2|-|a_2||a_3a_6-a_4a_5|+|a_3||a_3a_5-a_4^2|,$

$$|t_2|=\frac{-60\alpha}{31+90u}-\frac{229}{1728}, \qquad (22)$$

$|t_3|=|a_3a_6-a_4a_5|+|a_2||a_2a_6-a_3a_5|-|a_4||a_2a_4-a_3^2|,$

$$|t_3|=\frac{378\alpha}{31+90u}-\frac{6115}{5184}, \qquad (23)$$

$|t_4|=|a_3||a_2a_4-a_2^2|-|a_4||a_4-a_2a_3|+|a_5||a_3-a_2^2|,$

$$|t_4|=-\frac{667}{1296}. \qquad (24)$$

Inserting values $(21)-(24)$ in (5), we obtain

$$|H_4(1)| \leq -\Upsilon_1(\delta,u)+\Upsilon_2(\delta,u)-\Upsilon_3(\delta,u)+\Upsilon_4(\delta,u),$$

where

$$\Upsilon_1(\delta,u)=\frac{2533\alpha}{24(31\ +\ 90u)}-\frac{1271447}{1679616},$$

$$\Upsilon_2(\delta,u)=-60*\frac{-60\alpha}{31\ +\ 90u}-\frac{229}{1728},$$

$$\Upsilon_3(\delta,u)=81648*\frac{81648\alpha^2}{(31\ +\ 90u)^2}-\frac{6115\alpha}{24(31\ +\ 90u)},$$

$$\Upsilon_4(\delta,u)=\frac{-19343\alpha}{36(43\ +\ 168u)}.$$

In case $\mu=0$, we get the following Corollary.

Corollary 3: If function f of form (1) belongs to the subclass $\mathcal{N}(\alpha,\mu,e)$, then

$$|H_4(1)| \leq -\Theta_1(\delta,u)+\Theta_2(\delta,u)-\Theta_3(\delta,u)+\Theta_4(\delta,u),$$

where

$$\Theta_1(\delta,u)=\frac{2533\alpha}{744}-\frac{1271447}{1679616}, \qquad \Theta_2(\delta,u)=-\frac{60\alpha}{31}-\frac{229}{1728},$$

$$\Theta_3(\delta,u)=\frac{81648\alpha^2}{961}-\frac{6115\alpha}{744}, \qquad \Theta_4(\delta,u)=-\frac{19343\alpha}{1548}.$$

In case $\alpha=1\ and\ \mu=0$, we get the following Corollary.

Corollary 4: If function f of form (1) belongs to the subclass $\mathcal{N}(\alpha,\mu,e)$, then

$$|H_4(1)| \leq\approx -93.95348065.$$

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