Applications of the $PG(6,2)$ in Coding Theory

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Abstract

In this research paper, we examine how the projective space $PG(6,2)$ over the Galois field, contributes to the construction and analysis of projective codes denoted as $[n,k,d]\_q$. The fundamental parameters of these codes are defined, where n represents the code length, represents the minimum distance. Furthermore, we $d$represents the code dimension, and k calculate the error correction capacity e based on the drop matrix, which plays a critical role in the structural properties of the code. Additionally, we demonstrate that this code is optimal.

Keywords: projective space, incidence matrix, Hamming bound, e-error correcting, coding theory

1. Introduction

When mentioning the coding theory, one must look at what many mathematicians have done, specifically studying the application of projective space to the Kahlua field, where many theories and definitions have emerged about the relationships between finite projective geometry and the coding theory. For example, Hirschfeld [1] showed some theories and definitions about the relationships between finite projective geometry and coding theory. Finite projective geometry and coding theory. Al-Saraji [2, 3] also translated the links between the projective level of order 17 and error correction codes and presented some other important results.

Hill [4] presented the concepts and tools of coding theory. Al-Zangana [5, 6, 7, 8] discussed the relationship between the projective level of order 19 and error-correcting codes. Now we look at the projective space on a finite field of order q, where q=2

2.Construction of $PG(6,2)$

The polynomial of degree seven $f(x) = x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - 1$ is primitive over $F_2 = \{0,1\}$, since , $f(0) = 1$, $f(1) = 1$

Theorem 2.1 : [6] sphere packing or Hamming bound

$A\ q - ary(n, M, 2e + 1) - code\ C$ satisfies

$$M\left\{\binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e}(q-1)^e\right\} \leq q^n$$

Corollary2.2 [2]:A q-ary (n.M,2e+1) code C is perfect if and only if equality holds in Theorem 2.1

Definition 2.3 [5]: A q-ary code C of length n is a subset of (F_q)

Definition 2.4 Linear Codes [9] : The minimum distance d of a non-trivial code C is given by

$$d = min\{d(x,y) | x \in C, y \in C, x \neq y\}$$

$$P_i = [1,0,0,0,0,0,0]\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{i-1}, i = 1,2,\ldots,127,$$

Table (2. 1): The points of $PG(6,2)$ are:

$\mathcal{P}_1$	(1,0,0,0,0,0,0)	$\mathcal{P}_{51}$	(1,0,0,1,1,0,1)	$\mathcal{P}_{101}$	(0,1,1,0,1,1,0)
$\mathcal{P}_2$	(0,1,0,0,0,0,0)	$\mathcal{P}_{52}$	(1,1,1,1,0,0,1)	$\mathcal{P}_{102}$	(0,0,1,1,0,1,1)
$\mathcal{P}_3$	(0,0,1,0,0,0,0)	$\mathcal{P}_{53}$	(1,1,0,0,0,1,1)	$\mathcal{P}_{103}$	(1,0,1,0,0,1,0)
⋮	...	⋮	...	⋮	...
$\mathcal{P}_{23}$	(1,1,0,1,1,0,1)	$\mathcal{P}_{73}$	(1,0,0,1,1,1,1)	$\mathcal{P}_{125}$	(1,0,0,0,0,1,1)
$\mathcal{P}_{24}$	(1,1,0,1,0,0,1)	$\mathcal{P}_{74}$	(1,1,1,1,0,0,0)	$\mathcal{P}_{126}$	(1,1,1,1,1,1,0)
$\mathcal{P}_{25}$	(1,1,0,1,0,1,1)	$\mathcal{P}_{73}$	(0,1,1,1,1,0,0)	$\mathcal{P}_{127}$	(0,1,1,1,1,1,1)

With selecting the point in $PG(6,2)$ such that the third cord innate equal to zero this means it belongs to l_0=$V(Z)$ such that all in $F_2 \setminus \{0\}$; therefore, p_i=i=1,2,…… ,127 we obtain $\mathfrak{L}_1 = \{1,2,110\}$ where

$$\mathfrak{L}_i = \mathfrak{L}_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{i-1} ,i = 1,2,\dots,127,$$

Table (2. 2): The lines of $PG(6,2)$ are:

$\mathfrak{L}_1$	1,2,110	$\mathfrak{L}_{51}$	51,52,33	$\mathfrak{L}_{101}$	101,102,83
$\mathfrak{L}_2$	2,3,111	$\mathfrak{L}_{52}$	52,53,34	$\mathfrak{L}_{102}$	102,103,84
$\mathfrak{L}_3$	3,4,112	$\mathfrak{L}_{53}$	53,54,35	$\mathfrak{L}_{103}$	103,104,85
⋮	...	⋮	...	⋮	...
$\mathfrak{L}_{23}$	23,24,5	$\mathfrak{L}_{73}$	73,74,55	$\mathfrak{L}_{125}$	125,126,107
$\mathfrak{L}_{24}$	24,25,6	$\mathfrak{L}_{74}$	74,75,56	$\mathfrak{L}_{126}$	126,127,108
$\mathfrak{L}_{25}$	25,267	$\mathfrak{L}_{75}$	75,76,57	$\mathfrak{L}_{127}$	127,1,109

With selecting the point in $PG(6,2)$ such that the forth cord innate equal to zero this means it belongs to $plane_1$=$V(Z)$ such that all in $F_2\setminus\{0\}$; therefore, p_i=i=1,2,…… ,127 we obtain $plane_1 = \{1,2,3,9,92,110,111\}$ where

$$plane_i = plane_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{i-1} ,i = 1,2,\dots,127,$$

Table (2. 3): The planes of $PG(6,2)$ are:

π_1	1,2,3,9,92,110,111	π_{51}	51,52,53,59,15,33,34	π_{101}	101,102,103,109,65,83,84
π_2	2,3,4,10,93,111,112	π_{52}	52,53,54,60,16,34,35	π_{102}	102,103,104,110,66,84,85
π_3	3,4,5,11,94,112,113	π_{53}	53,54,55,61,17,35,36	π_{103}	103,104,105,111,67,85,86
⋮	...	⋮	...	⋮	...
π_{23}	23,24,25,31,114,5,6	π_{73}	73,74,75,81,37,55,56	π_{125}	125,126,127,6,89,107,108
π_{24}	24,25,26,32,115,6,7	π_{74}	74,75,76,82,38,56,57	π_{126}	126,127,1,7,90,108,109
π_{25}	25,26,27,33,116,7,8	π_{75}	75,76,77,83,39,57,58	π_{127}	127,1,2,8,91,109,110

With selecting the point in $\mathrm{PG}(6,2)$ such that the five cord innate equal to zero this means it belongs to $SUBSPACE_1$=$V(Z)$ such that all in $F_2\backslash\{0\}$; therefore, p_i=i=1,2,…… ,127 we obtain $SUBSPACE_1$ ={1,2,3,4,9,10,32,74,78,92,93,110,111,112,118} where

$$SUBSPACE_i = SUBSPACE_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{i-1} ,i = 1,2, \dots ,127,$$

Table (2. 4): The spaces of $\mathrm{PG}(6,2)$ are:

$subspace_1$	1,2,3,4,9,10,32,74,78,92,93,110,111,112,118
$subspace_2$	2,3,4,5,10,11,33,75,79,93,94, 111,112,113,119
$subspace_3$	3,4,5,6,11,12,34,76,80,94,95, 112,113,114,120
⋮	…
$subspace_{125}$	125,126,127,1,6,7,29,71,75,89, 90,107,108,109,115
$subspace_{126}$	126,127,1,2,7,8,30,72,76,90,91,108,109,110,116
$subspace_{127}$	127,1,2,3,8,9,31,73,77,91,92, 109,110,111,117

With selecting the point in $\mathrm{PG}(6,2)$ such that the six cord innate equal to zero this means it belongs to $PG(4,2)_1$=$V(Z)$ such that all in $F_2\backslash\{0\}$; therefore, p_i=i=1,2,…… ,127 we obtain $PG(4,2)_1$ ={1,2,3,4,5,9,10,11,14,17,32,33,44,56,60,65,74,75,78,79,89,92,93,94,100,110,111,112 ,113,118,119} where

$$PG(4,2)_i = PG(4,2)_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{i-1} ,i = 1,2, \dots ,127,$$

Table (2. 5): The $\mathrm{PG}(4,2)$ of $\mathrm{PG}(6,2)$ are:

$PG(4,2)_1$	1,2,3,4,5,9,10,11,14,17,32,33,44,56,60,65,74,75,78,79,89,92,93,94,100,110,111,112,113,118,119
$PG(4,2)_2$	2,3,4,5,6,10,11,12,15,18,33,34,45,57,61,66,75,76,79,80,90,93,94,95,101,111,112,113,114,119,120
$PG(4,2)_3$	3,4,5,6,7,11,12,13,16,19,34,35,46,58,62,67,76,77,80,81,91,94,95,96,102,112,113,114,115,120,121
…	⋮

$PG(4,2)_{125}$	125,125,126,127,1,2,6,7,8,11,14,29,30,41,53,57,62,71,72,75,76,86,89,90,91,97,107,108,109,110,115,116
$PG(4,2)_{126}$	126,126,127,1,2,3,7,8,9,12,15,30,31,42,54,58,63,72,73,76,77,87,90,9192,98,108,10,110,111,116,117
$PG(4,2)_{127}$	127,127,1,2,3,4,8,9,10,13,16,31,32,43,55,59,64,73,74,77,78,88,9,92,93,99,109,110,111,112,117,118

With selecting the point in $PG(6,2)$ such that the seven cord innate equal to zero this means it belongs to $PG(5,2)_1$=$V(Z)$ such that all in $F_2\setminus\{0\}$; therefore, p_i=i=1,2,…… ,127 we obtain $PG(5,2)_1$ ={1,2,3,4,5,6,9,10,11,12,14,15,17,18,26,32,33,34,38,40,42,44,45,47,49,54,56,57,60,61,65,66,69,71,74,75,76,78,79,80,82,86,89,90,92,93,94,95,100,101,103,106,110,111,112,113,114,116,118,119,120,123,126} where

$$PG(5,2)_i = PG(5,2)_1 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{i-1} ,i = 1,2, \ldots,127,$$

Table (2. 6): The $PG(5,2)$ of $PG(6,2)$ are:

$PG(5,2)_1$	1,2,3,4,5,6,9,10,11,12,14,15,17,18,26,32,33,34,38,40,42,44,45,47,49,54,56,57,60,61,65,66,69,71,74,75,76,78,79,80,82,86,89,90,92,93,94,95,100,101,103,106,110,111,112,113,114,116,118,119,120,123,126
$PG(5,2)_2$	2,3,4,5,6,7,10,11,12,13,15,16,18,19,27,33,34,35,39,41,43,45,46,48,50,55,57,58,61,62,66,67,70,72,75,76,77,79,80,81,83,87,90,91,93,94,95,96,101,102,104,107,111,112,113,114,115,117,119,120,121,124,127
$PG(5,2)_3$	3,4,56,7,8,11,12,13,14,16,17,19,20,28,34,35,36,40,42,44,46,47,49,51,56,58,59,62,63,67,68,71,73,76,77,78,80,81,82,84,88,91,92,94,95,96,97,102,103,105,108,112,113,114,115,116,118,120,121,122,125,1
…	⋮
$PG(5,2)_{125}$	125,126,127,1,2,3,6,7,8,9,11,12,14,15,23,29,30,31,35,37,39,41,42,44,46,51,53,54,57,58,62,63,66,68,71,72,73,75,76,77,79,83,86,87,89,90,91,92,97,98,100,103,107,108,109,110,111,113,115,116,117,120,123
$PG(5,2)_{126}$	126,127,1,2,3,4,7,8,9,10,12,13,15,16,24,30,31,32,36,38,40,42,43,45,47,52,54,55,58,59,63,64,67,69,72,73,74,76,77,78,80,84,87,88,90,91,92,93,98,99,101,104,108,109,110,111,112,114,116,117,118,121,124
$PG(5,2)_{127}$	127,1,2,3,4,5,8,9,10,11,13,14,16,17,25,31,32,33,37,39,41,43,44,46,48,53,55,56,59,60,64,65,68,70,73,74,75,77,78,79,81,85,88,89,91,92,93,94,99,100,102,105,109,110,111,112,113,115,117,118,119,122,125

3-New Results :

in the following theorem the parameters n,m,d are constructed.

Theorem 3,1 : The projective space of order 2 is a code with a parameter $[\, n = 127, M = 2^8, d = 63 \,]$.

The $PG(5,2)$ has an incidence matrix $A = (a_{ij})$ where

$a_{ij} = \begin{cases} 1 \; if \; p_i \in PG(5,2) \\ 0 \; if \; p_i \notin PG(5,2) \end{cases}$,Then we have the incider Table (3,1).

Table (3,1)

p_i / $PG(5,2)_i$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	...	p_{125}	p_{126}	p_{127}
$PG(5,2)_1$	1	1	1	1	1	1	0	0	1	1	...	0	1	0
$PG(5,2)_2$	0	1	1	1	1	1	1	0	0	1	...	0	0	1
$PG(5,2)_3$	1	0	1	1	1	1	1	1	0	0	...	1	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	...	⋮	⋮	⋮
$PG(5,2)_{125}$	1	1	1	0	0	1	1	1	1	0	...	1	1	1
$PG(5,2)_{126}$	1	1	1	1	0	0	1	1	1	1	...	0	1	1
$PG(5,2)_{127}$	1	1	1	1	1	0	0	1	1	1	...	1	0	1

Galois field element of the 2 order so i= 1,2,3,…….,127

$\varpi_i = [0,0,0,0,0,0,0, \dots \dots \dots \dots \dots \dots \dots \dots \dots, 0,0,0,0,]$

$\sigma_i = [1,1,1,1,1,1,1, \dots \dots \dots \dots \dots \dots \dots \dots \dots, 1,1,1,1,]$

Combine σ_i with a binary system projective matrix we get a zero-one

$matrix \;\; \sigma_i = [1,1,1,1,1,1,1, \dots \dots \dots \dots \dots \dots \dots \dots \dots, 1,1,1,1,]$

Let $k_i = \sigma_i + \mathrm{PG}(5,2)_i \quad 1 \leq \; i \; \leq 127$

Table (3, 2) Now the table of $\mathscr{k}_i$

p_i / $\mathscr{k}_i$	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	...	p_{125}	p_{126}	p_{127}
$\mathscr{k}_1$	0	0	0	0	0	0	1	1	0	0	...	1	0	1
$\mathscr{k}_2$	1	0	0	0	0	0	0	1	1	0	...	1	1	0
$\mathscr{k}_3$	0	1	0	0	0	0	0	0	1	1	...	0	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	...	⋮	⋮	⋮
$\mathscr{k}_{125}$	0	0	0	1	1	0	0	0	0	1	...	0	0	0
$\mathscr{k}_{126}$	0	0	0	0	1	1	0	0	0	0	...	1	0	0
$\mathscr{k}_{127}$	0	0	0	0	0	1	1	0	0	0	...	0	1	0

Find the shortest distance between two different code word in the matrices above, so that the shortest distance is 63 and largest distance is 127

Table (3,4): the minimum distance d of a non-trivial code C

$d(\varpi, pg(5,2)_i) = 63$	$d(pg(5,2)_i, \mathscr{k}_i) = 127$	$d(\varpi_i, \sigma_i) = 127$
$d(\sigma, pg(5,2)_i) = 64$	$d(\varpi_i, \mathscr{k}_i) = 64$	

If we substitute the values of n = 127, d = 63, e = 31 in inequality of Theorem 3.1 we get M= 7396 Hence C is a $(127, 2^8 6, 63) - code.$

$$M\left\{\binom{n}{0} + \binom{n}{1}(q-1) + \cdots + \binom{n}{e}(q-1)^e\right\} \le q^n$$

As $n = q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$

And $M = q^{k+1}$

$$2^8\left\{\begin{matrix} \binom{127}{0} + \binom{127}{1}(2-1) + \binom{127}{2}(2-1)^2 + \binom{127}{3}(2-1)^3 + \binom{127}{4}(2-1)^4 \\ +\binom{127}{5}(2-1)^5 + \cdots + \cdots + \binom{127}{27}(2-1)^{27} + \binom{127}{28}(2-1)^{28} \\ +\binom{127}{29}(2-1)^{29} + \binom{127}{30}(2-1)^{30} + \binom{127}{31}(2-1)^{31} + \end{matrix}\right\} \le 2^{127}$$

Via Corollary 2.2, therefore C is perfect.

CONCLUSIONS

We have constructed the Projective linear codes with the parameters n, k, and d depending on the order of Galois Field F_2 , and have also studied the relationship between the finite Projective space and coding theory such that the columns of the generator matrix of any linear code are considered the points in the Projective spac

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