

Solving Harmonic Equation by Implicit Method

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Abstract:

In this paper, we solving the harmonic equation by one of the differencing method which is the implicit numerical method the heat diffusion equation is one kinds of homogeneous harmonic equation. So, we solving this equation by this numerical method with the initial and boundary conditions. The results other method.

Keywords: Harmonic, Heat Diffusion Equation Numerical Method, Implicit Method.

1-1 Introduction:

The Partial Differential Equations (P. d. e) are used to expressing for many problems in many different Sciences, Fluid Mechanic, Physics, Chemical, Engineering, Elasticity, ... etc.

One kinds of (P. d. e) is the Homogeneous Harmonic Equation, the general formula of this equation is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \dots (1 - 1)$$

Or as the form

$$\nabla^2 v = 0 \quad \dots (1 - 2)$$

The heat diffusion equation is one kind of this equation many researchers are solve this equation by the Numerical Method, in (2020), Adak is solving the general diffusion by Implicit and Explicit method [1] and Awni M. Gaftan in 2019 is used the θ – differencing

method for testing the effect the Isolating of some materials [2]. In (2004), Recktenwald is introduced the finite – differenced approximations to the heat equation [5] and in (2013), Thomas, J. W. is introduced the Numerical Partial differential equation with boundary –value problems [7].

1-2 The Implicit Method: [1] [2]

Is a numerical way of expressing the value of $v(x)$ in the future time $(t+\Delta t)$, depending on the future time $(t + \Delta t)$ and the present time (t) , and sometimes also about the past tense $(t-\Delta t)$.

The division is according to the explicit method into levels. After defining the initial conditions and boundary conditions, each level is divided into points.

They represent initial (primary) and boundary conditions.

We use the following equations to find the points in the levels:

- 1- The general form of the implicit method for the one-dimensional heat flow equation.

• One - dimension.

$$v_{i,j} = (1 + 2R)v_{i,j+1} - Rv_{i-1,j+1} - Rv_{i+1,j+1} \quad \dots (1 - 3)$$

whereas: - $\forall i = 1,2,3, \dots, n$, $\forall j = 0,1,2,3, \dots, m$

$$R = \frac{k\Delta t}{h^2} , (\Delta x)^2 = h^2$$

- 2- The general form of the implicit method for solving the homogeneous harmonic equation.

$$v_{i-1,j+1} + v_{i+1,j-1} - 2v_{i+1,j} = 2v_{i,j+1} - 2v_{i+1,j+1} \quad \dots (1 - 4)$$

$\forall i = 1,2,3, \dots, n$, $\forall j = 0,1,2,3, \dots, m$

And we express it as $v_{i,j+1}$ by $v_{i,j}$, we find that the time domain and the spatial domain are already represented in each time period, we write (j) (inner node), and the equations are solved simultaneously with the initial conditions and borderline

So the implicit method is easily as shown in the following figure [1], [2].

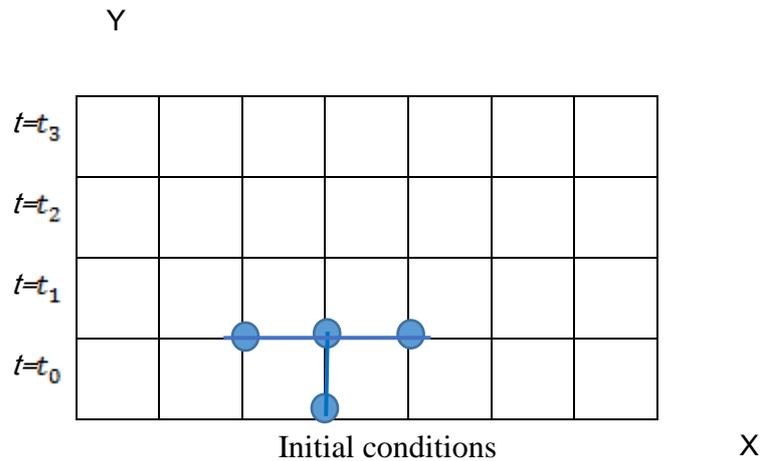


Figure (1 – 1)

1-3 Algorithm of Implicit Method:

1- We divide the solution area into several divisions that represent each level and are in the form of a rectangular grid, and we place the initial conditions, the right boundary conditions, and the left boundary conditions mentioned in the problem.

2 - Substitute values ($\forall i = 1, 2, 3, \dots, n, j = 0$) into the general form of the equations, the implicit method (1 – 3)

3 - Substituting the values of the boundary conditions, the values of the initial conditions, and the value of R in the general formula, so we will get three general equations simultaneously with the addition of substituting the values of the initial conditions and the values of the boundary conditions.

4 - After the substitutions made in step (3) and solving the three equations, we get the unknown values ($v_{3,1}, v_{2,1}, v_{1,1}$) and by repeating the process on the previous steps when $j=1$.

5- Continuing to repeat the process in order to get the approximate solution that satisfies the condition.

1-4 Initial conditions: [5]

In order to solve a partial differential equation of the second degree, it is required to know the initial conditions of any harmonic and bi-harmonic equation. When we solve problems that appear in the harmonic and bi-harmonic equations, we have to know the initial conditions of the wave equation, which depends on two functions g and f as linear sinusoidal functions or as complex exponential functions

$$v(x, t)_{t=0} = f(x, 0)$$

$$v(x, t) = v(x) \quad , \quad 0 < x < l$$

1-5 Boundary conditions and their types: [3] [8]

1- The boundary conditions of the harmonic and bi-harmonic equation consist of a set of types as follows

The boundary conditions of the harmonic and bi-harmonic equations are met according to the following mathematical formulas

$$v(t, 0) = v_1(t) \quad , \quad 0 < x < l$$

$$v(t, l) = v_2(t) \quad , \quad 0 < x < l$$

This boundary condition is called the (Dirichlet Condition) in which the value of the function is known through each boundary point on the boundary of the solution region.

2 - While the boundary condition in the harmonic and bi-harmonic equation allocated within the borders of any region is in the following form

$$c \frac{\partial v}{\partial x}(t, 0) = 0$$

It is called (Neumann Condition), where the value of the derivative of the used function is clear through every boundary point

3 - Robin Condition within the limits is in the form

$$c \frac{\partial v}{\partial x} + hv(t, l) = g(t)$$

And this condition depends on the partial derivative of the function with the variable x

In conclusion, the three conditions that were mentioned earlier are used during the boundaries of the region by relying on knowing the value of the function once and knowing the value of its derivative within each region or on part of the boundaries

All conditions can be mixed, so we can get some values of the function or its derivatives through some boundary points. Therefore, the boundary conditions must be compensated once for the function itself and once for its derivatives.

1-6 Deriving the general formula for the implicit method for solving the one-dimensional heat equation: [2] [4] [7]

The one-dimensional heat equation is

$$\frac{\partial^2 v}{\partial x^2} = \left[\frac{1}{k} \right] \frac{\partial v}{\partial t} \quad \dots (1 - 5)$$

From equation (3 - 1) we have

$$v_{xx} = \frac{\partial^2 v}{\partial x^2}, \quad v_t = \frac{\partial v}{\partial t}$$

In order to get the general form of the implicit method, we use the finite differences

$$v_{xx} = \frac{v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}}{(\Delta x)^2} \quad \dots (1 - 6)$$

$$v_t = \frac{v_{i,j+1} - v_{i,j}}{\Delta t} \quad \dots (1 - 7)$$

$$\forall i = 1, 2, 3, \dots, n, \quad \forall j = 0, 1, 2, \dots, m$$

Where we substitute the differences ending (1 - 6) and (1 - 7) in equation (1 - 5) we get

$$\frac{v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}}{h^2} = \frac{1}{k} \left(\frac{v_{i,j+1} - v_{i,j}}{\Delta t} \right) \quad \dots (1 - 8)$$

We multiply both sides of equation (1 - 8) by (k Δt) we get

$$\frac{k\Delta t}{h^2} (v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}) = v_{i,j+1} - v_{i,j} \quad \dots (1 - 9)$$

Suppose that

$$R = \frac{k\Delta t}{h^2}$$

So equation (1 - 9) becomes as follows

$$R(v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}) = v_{i,j+1} - v_{i,j} \quad \dots (1 - 10)$$

After simplifying equation (1 - 10) to its simplest form, we get the general formula for the implicit method, which is:

$$v_{i,j} = (1 + 2R)v_{i,j+1} - Rv_{i-1,j+1} - Rv_{i+1,j+1} \quad \dots (1 - 11)$$

$$\forall i = 1,2,3, \dots, n, \quad \forall j = 0,1,2, \dots, m$$

Whereas:

$$R = \frac{k\Delta t}{h^2}$$

1-7 Deriving the implicit method for the harmonic equation:

It is called the implicit method, since the solution for each level we walk through requires it to solve a system of linear equations with a dimension $(n \times n)$ and the points required to find the solution through the $j+1$ level are based on a known point in the level, and to find the point $v_{i,j+1}$ at the level we do the following: -

1- The general form of the homogeneous harmonic equation is:

$$v_{xx} + v_{yy} = 0 \quad \dots (1 - 12)$$

1- We use the following finite differences [5]

$$v_{xx} = \frac{v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}}{(\Delta x)^2} \quad \dots (1 - 8)$$

$$v_{yy} = \frac{v_{i+1,j-1} - 2v_{i+1,j} + v_{i+1,j+1}}{(\Delta y)^2} \quad \dots (1 - 13)$$

$$\forall i = 1,2,3, \dots, n, \quad \forall j = 0,1,2,3, \dots, m$$

knowing that: $(\Delta x)^2 = (\Delta y)^2 = h^2$

2- We substitute equations (1 - 6) and (1 - 13) into equation (1 - 12) as shown below

$$\frac{v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}}{h^2} + \frac{v_{i+1,j-1} - 2v_{i+1,j} + v_{i+1,j+1}}{h^2} = 0 \quad \dots (1 - 14)$$

By multiplying equation (1 - 14) by (h^2) for both sides and simplifying the equation, we get

$$v_{i-1,j+1} - 2v_{i,j+1} + 2v_{i+1,j+1} + v_{i+1,j-1} - 2v_{i+1,j} = 0 \quad \dots (1 - 15)$$

And it can be formulated as follows as well

$$v_{i-1,j+1} + v_{i+1,j-1} - 2v_{i+1,j} = 2v_{i,j+1} - 2v_{i+1,j+1} \quad \dots (1 - 16)$$

$$\forall i = 1,2,3, \dots, n, \quad \forall j = 0,1,2,3, \dots, m$$

Whereas, equation (1 – 16) is the general form of the implicit method for solving the homogeneous harmonic equation.

Where the solution area is divided into levels and each level contains points that represent the initial conditions and the boundary conditions, where equation (1 – 16) is used to find the points in the levels and are solved simultaneously and with the frontier and initial conditions.

Example: - Use the implicit method to solve the one-dimensional heat flow equation

$v_t = v_{xx}$ on a rectangular plate, its dimensions are 4 * 2, applied on the x-axis and the y-axis at the center of the origin, according to the following data:

$$\Delta t = 4 \text{ sec} \quad , \quad \Delta x = 2 \quad , \quad k = 0.5$$

and primary conditions

$$v(x, t) = 0^\circ \quad , \quad 0 < x < 12 \quad , \quad t = 0$$

and boundary conditions

$$v(0, t) = 5^\circ \quad , \quad v(10, t) = 3^\circ$$

Solution / We initially make a grid that contains nodal points and put the data in the issue in it and it is in the form

We write the general equation for the implicit method in solving the one-dimensional heat flow equation, namely

$$v_{i,j} = (1 + 2R)v_{i,j+1} - Rv_{i-1,j+1} - Rv_{i+1,j+1} \quad \dots (1 - 11)$$

whereas: - $\forall i = 1, 2, 3, \dots, n \quad , \quad \forall j = 0, 1, 2, 3, \dots, m$

We find the value of R through the following mathematical formula

$$R = \frac{k\Delta t}{h^2} \quad , \quad (\Delta x)^2 = h^2$$

$$R = 0.5 \frac{4}{(2)^2} \quad \Rightarrow \quad R = 0.5$$

We start when $i = 1, 2, 3, 4, 5 \quad , \quad j = 0$

$$i = 1 \quad , \quad j = 0$$

$$v_{1,0} = (1 + 2R)v_{1,1} - Rv_{0,1} - Rv_{2,1} \quad \dots (1 - 17)$$

$$i = 2 , \quad j = 0$$

$$v_{2,0} = (1 + 2R)v_{2,1} - Rv_{1,1} - Rv_{3,1} \quad \dots (1 - 18)$$

$$i = 3 , \quad j = 0$$

$$v_{3,0} = (1 + 2R)v_{3,1} - Rv_{2,1} - Rv_{4,1} \quad \dots (1 - 19)$$

$$i = 4 , \quad j = 0$$

$$v_{4,0} = (1 + 2R)v_{4,1} - Rv_{3,1} - Rv_{5,1} \quad \dots (1 - 20)$$

$$i = 5 , \quad j = 0$$

$$v_{5,0} = (1 + 2R)v_{5,1} - Rv_{4,1} - Rv_{6,1} \quad \dots (1 - 21)$$

We substitute the value of $R = 0.5$ and the values of the boundary conditions and initial conditions into equations (1 - 17), (1 - 18), (1 - 19), (1 - 20) and (1 - 21), and then we simplify and arrange the equations, we get

$$2.5 = 2v_{1,1} - 0.5v_{2,1} \quad \dots (1 - 22)$$

$$0 = -0.5v_{1,1} + 2v_{2,1} - 0.5v_{3,1} \quad \dots (1 - 23)$$

$$0 = -0.5v_{2,1} + 2v_{3,1} - 0.5v_{4,1} \quad \dots (1 - 24)$$

$$0 = -0.5v_{3,1} + 2v_{4,1} - 0.5v_{5,1} \quad \dots (1 - 25)$$

$$1.5 = -0.5v_{4,1} + 2v_{5,1} \quad \dots (1 - 26)$$

By multiplying equation (1 - 23) by (4), we get

$$0 = -2v_{1,1} + 8v_{2,1} - 2v_{3,1} \quad \dots (1 - 27)$$

We take the two equations (1 - 23) and (1 - 27) and solve them simultaneously (by the combination), so we get equation (1 - 28)

$$2.5 = 2v_{1,1} - 0.5v_{2,1} \quad \dots (1 - 23)$$

$$0 = -2v_{1,1} + 8v_{2,1} - 2v_{3,1} \quad \dots (1 - 27)$$

$$2.5 = 7.5v_{2,1} - 2v_{3,1} \quad \dots (1 - 28)$$

By solving equations (1 – 24) and (1 – 28) simultaneously (by the combination), we get equation (1 – 29)

$$0 = -0.5v_{2,1} + 2v_{3,1} - 0.5v_{4,1} \quad \dots (1 - 24)$$

$$2.5 = 7.5v_{2,1} - 2v_{3,1} \quad \dots (1 - 28)$$

$$2.5 = 7v_{2,1} - 0.5v_{4,1} \quad \dots (1 - 29)$$

By multiplying equation (1 – 29) by (4), we get

$$10 = 28v_{2,1} - 2v_{4,1} \quad \dots (1 - 30)$$

By solving equations (1 – 25) and (1 – 30) simultaneously (by the combination), I get equation (1 – 31)

$$0 = -0.5v_{3,1} + 2v_{4,1} - 0.5v_{5,1} \quad \dots (1 - 25)$$

$$10 = 28v_{2,1} - 2v_{4,1} \quad \dots (1 - 30)$$

$$10 = 28v_{2,1} - 0.5v_{3,1} - 0.5v_{5,1} \quad \dots (1 - 31)$$

By multiplying equation (1 – 31) by (4), I get:

$$40 = 112v_{2,1} - 2v_{3,1} - 2v_{5,1} \quad \dots (1 - 32)$$

We take equations (1 – 26) and (1 – 32) and solve them simultaneously (by the combination) to obtain equation (1 – 33)

$$1.5 = -0.5v_{4,1} + 2v_{5,1} \quad \dots (1 - 26)$$

$$40 = 112v_{2,1} - 2v_{3,1} - 2v_{5,1} \quad \dots (1 - 32)$$

$$41.5 = 112v_{2,1} - 4v_{3,1} - 0.5v_{4,1} \quad \dots (1 - 33)$$

By multiplying equation (1 – 33) by (4), we get:

$$166 = 448v_{2,1} - 16v_{3,1} - 2v_{4,1} \quad \dots (1 - 34)$$

By solving equations (1 – 29) and (1 – 34) simultaneously (by subtraction), we get equation (1 – 35)

$$2.5 = 7v_{2,1} - 0.5v_{4,1} \quad \dots (1 - 29)$$

$$166 = 448v_{2,1} - 16v_{3,1} - 2v_{4,1} \quad \dots (1 - 34)$$

$$156 = 420v_{2,1} - 16v_{3,1} \quad \dots (1 - 35)$$

By multiplying equation (1 – 28) by (8), we get

$$20 = 60v_{2,1} - 16v_{3,1} \quad \dots (1 - 36)$$

By solving equations (1 – 35) and (1 – 36) simultaneously (by subtraction), we get

$$156 = 420v_{2,1} - 16v_{3,1} \quad \dots (1 - 35)$$

$$20 = 60v_{2,1} - 16v_{3,1} \quad \dots (1 - 36)$$

$$136 = 360v_{2,1}$$

$$\therefore v_{2,1} = 0.3777$$

Substitute the value of $v_{2,1} = 0.3777$ into equation (1 – 35)

$$156 = 420(0.3777) - 16v_{3,1}$$

$$16v_{3,1} = 2.6634$$

$$\therefore v_{3,1} = 0.1665$$

Substituting the value of $v_{2,1} = 0.3777$ into equation (1 – 22), we get

$$2.5 = 2v_{1,1} - 0.5(0.3777)$$

$$2v_{1,1} = 2.6885$$

$$\therefore v_{1,1} = 1.3444$$

Substitute the value of $v_{2,1} = 0.3777$ into equation (1 – 29)

$$2.5 = 7(0.3777) - 0.5v_{4,1}$$

$$0.5v_{4,1} = 0.1439$$

$$\therefore v_{4,1} = 0.2878$$

Substituting the value of $v_{4,1} = 0.2878$ into equation (1 – 26), we get the following

$$0 = -0.5(0.2878) + 2v_{5,1} - 1.5$$

$$2v_{5,1} = 1.6439$$

$$\therefore v_{5,1} = 0.82195$$

In the same way as the solution when applied to the other levels, you will get the following point values.

$$v_{1,2} = 2.12284$$

$$v_{2,2} = 0.82557$$

$$v_{3,2} = 0.41829$$

$$v_{4,2} = 0.51458$$

$$v_{5,2} = 1.28962$$

$$v_{1,3} = 2.62751$$

$$v_{2,3} = 1.25287$$

$$v_{3,3} = 0.73283$$

$$v_{4,3} = 0.84186$$

$$v_{5,3} = 1.60528$$

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