On invo-t-clean graph over the ring of integers modulo *p*

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Abstract

In this paper, we introduce and study the involution-t-clean Graph $G_{itc}(Z_p)$ defined over the ring of integers modulo p, where p is a prime number. The vertex set of $G_{itc}(Z_p)$ is Z_p, and two distinct vertices x and y are adjacent if and only if $x+y \in itc(Z_p)$, where $itc(Z_p)$ denotes the set of invo-t-clean elements in Z_p. The $G_{itc}(Z_p)$ is connected, and each vertex has a degree of either four or five. Using this characterization, we investigate several fundamental graph-theoretic properties. Our main result determines the exact diameter of $G_{itc}(Z_p)$: $diam(G_{itc}(Z_p)) = \left\lfloor \frac{p+2}{4} \right\rfloor$.

Keywords: invo-t-clean ring, diameter, degree, ring of integers modulo p, tripotent.

1. Introduction:

Throughout this paper, we assume that all rings are associative with identity. We focus primarily on the ring of integers modulo p, denoted by Z_p, where p is a prime number. Additionally, we consider the matrix ring $M_2(Z_2)$, the triangular matrix ring $TM_2(Z_2)$, and the set of involution elements $invo(R) = \{u \in R: u^2 = 1\}$, the set of tripotent elements $Tri(R) = \{t \in R: t^3 = t\}$.

The concept of clean ring theory has attracted considerable attention in recent years, see [1], [2] and [3], leading to various generalizations and applications in both algebra and graph theory. It starts with Beck in [4]. Also, several studies take the clean ring see [5], [6] and [7]. Among these generalizations, the notion of involution-t-clean rings, see [8], two-involution clean rings see [9] and [10], weakly 2-invo-clean rings [3]. Recently, Ahmad in [8] introduced and studied the concept of invo-t-clean rings, where an element $a \in R$ is called invo-t-clean if it can be expressed as $a = u + t$, with $u \in invo(R)$ and $t \in Tri(R)$. They investigated various algebraic properties of such rings, particularly focusing on the ring of integers modulo n, matrix rings, and other classical ring constructions. In the same work, the authors also initiated the study of a graph structure associated with invo-t-clean rings. They defined the graph $Cl_t(R)$ whose vertex set consists of ordered pairs (u, t) where $u \in invo(R)$and $t \in Tri(R)$, with the additional condition that $u + t$ is an invo-t-clean element. Two vertices $h_1 = (u_1, t_1)$ and $h_2 = (u_2, t_2)$ are adjacent if either $u_1 + u_2 = 0$ or $t_1 \cdot t_2 = 0$. This graph provides a combinatorial framework for understanding the interplay between involutions and tripotents in rings.

However, our definition takes a different approach, focusing specifically on invo-t-clean elements themselves. In our construction, the vertex set is precisely the ring R, establishing a natural correspondence between the algebraic structure and its associated graph. Each vertex represents an element of the ring, and an edge connects two distinct vertices x and y precisely when their sum $x + y$ is an invo-t-clean element. This formulation directly captures the additive relationship between ring elements and the invo-t-clean property.

In this paper, we focus on the graph $G_{itc}(Z_p)$ defined over the ring of integers modulo p, where p is a prime number. We prove that these graphs are connected and that the degrees of their vertices range between three and four. Additionally, we determine the exact number of edges. The main result of this paper, which was nontrivial to obtain, is a general formula for computing the diameter of the graph as prime p varies: $\text{diam}(G_{itc}(Z_p)) = \left\lfloor\frac{p+2}{4}\right\rfloor$.

2. Properties of the invo-t-clean graph of the Ring.

Definition 2.1: [8]

An element $a \in R$ is called involution t-clean (for short invo-t-clean), which can be written as $a = u + t,\ u \in invo(R)$ and $t \in Tri(R)$. The set of all invo-t-clean elements in the ring R denoted by $itc(R)$. A ring R is called to be invo-t-clean if $itc(R) = R$. An invo-t-clean ring with $ut = tu$ is strongly invo-t-clean.

Examples 2.2:

1. The rings Z_4, Z_5 and Z_6 are invo-t-clean.
2. The ring Z_7 is not invo-t-clean.
3. $M_2(Z_2)$, the upper triangular matrices and $TM_2(Z_3)$ are invo-t-clean rings.

Definition 2.3: [8]

An graph of ring R has an invo-t-clean elements which denoted by $Cl_t(R)$ has a vertex set $\mathcal{V}(Cl_t(R)) = \{(u,t){:}\ u \in invo(R), t \in Tri(R)\}$ and has the edge set $\mathcal{F}(Cl_t(R)) = \{h_1h_2{:}\, h_1 = (u_1,t_1), h_2 = (u_2,t_2),\ u_1 + u_2 = 0$ or $t_1 \cdot t_2 = 0, u_i \in \text{invo}(R), t_i \in Tri(R),\ u_i + t_i$ is an invo-t-clean element $i = 1,2\}$.

Example 2.4:

For the ring Z_5, the ring is invo-t-clean, $invo(Z_5) = \{1,4\}, Tri(Z_5) = \{0,1,4\}$,

$\mathcal{V}(Cl_t(Z_5)) = \{[1,0],[1,1],[1,4],[4,0],[4,1],[4,4]\}$

$\mathcal{F}(Cl_t(Z_5)) = \{([1,0],[1,0]),([1,0],[4,0]),([1,1],[1,1]),([1,1],[4,4]),([1,4],[1,4]),$

$([1,4],[4,1]),([4,0],[1,0]),([4,0],[4,0]),([4,1],[1,4]),([4,1],[4,1]),([4,4],[1,1]),([4,4],[4,4])\}$

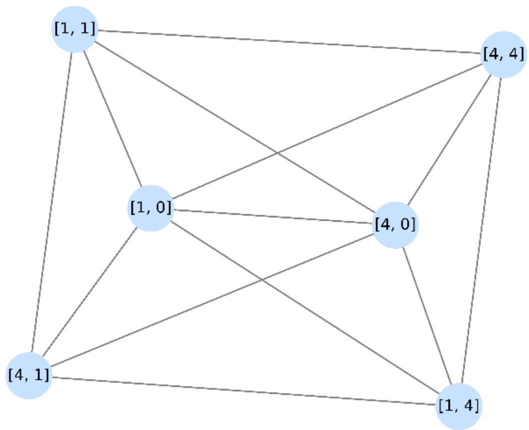

Figure 1: The Graph $Cl_t(Z_5)$

We now introduce the basic definition of the graph on rings, which constitutes the foundation of this paper.

Definition 2.5:

Let R be a ring. The invo-t-clean graph of R , denoted by $G_{itc}(R)$, is a graph with vertex set $V\big(G_{itc}(R)\big) = R$, and two distinct vertices x and y in $V\big(G_{itc}(R)\big)$ are adjacent if and only if $x + y \in itc(R)$.

We will focus our study on graphs defined over a single class of rings, namely fields. Specifically, we consider Z_p where p is a prime number, as illustrated in Examples 2.5, 2.6, and 2.7.

Example 2.5:

In the ring Z_5, $itc(Z_5) = Z_5$, Figure 2 illustrates the graph $G_{itc}(Z_5)$. In our representation, vertices having degree four are highlighted with a red border, whereas vertices corresponding to invo-t-clean elements are filled in orange.

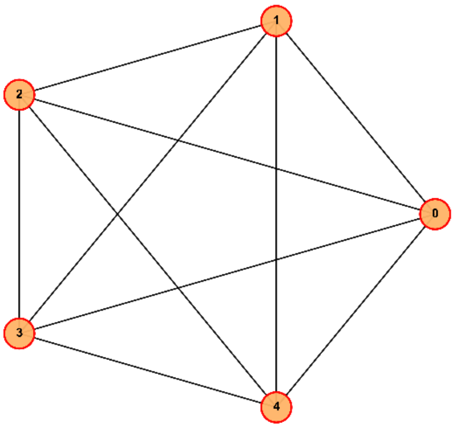

Figure 2: The Graph $G_{itc}(Z_5)$.

The fundamental contrast between the definition presented [8] and our current definition lies in the resulting graph structure. While the former relies on a fixed structure that renders graphs identical for all prime numbers (yet differs when n is the product of two primes), the representation in our current study varies according to the prime number in question. This is clearly illustrated in examples (2.5, 2.6, 2.7), where distinct prime values produced structurally non-identical graphs. Accordingly, this study focuses on investigating and identifying the common structural characteristics that unify these diverse graphs, in contrast to the approach adopted in the study [8].

Example 2.6:

In the ring Z_7, $itc(Z_7) = \{0,1,2,5,6\}$. Figure 3 illustrates the graph $G_{itc}(Z_7)$. In our representation, vertices having degree five are highlighted with a normal border, whereas vertices corresponding to non-invo-t-clean elements are filled in blue.

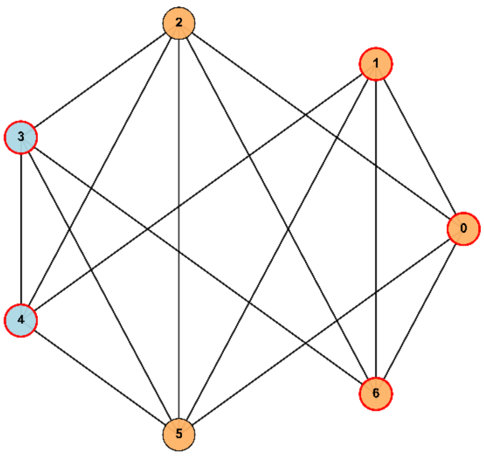

Figure 3: The Graph $G_{itc}(Z_7)$.

Example 2.7:

In the ring Z_{13}, $itc(Z_{13}) = \{0,1,2,10,11\}$. See Figure 4.

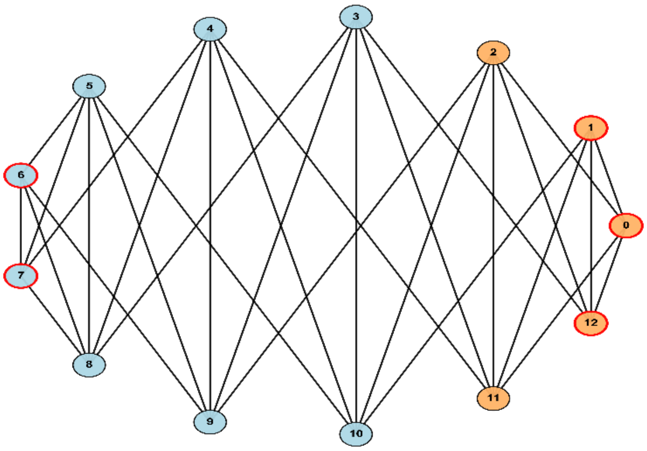

Figure 4: The Graph $G_{itc}(Z_{11})$.

Proposition 2.8

The ring Z_p where $p \geq 5$ is prime, the ring has

1. $invo(Z_p) = \{1, p-1\}$
2. $Tri(Z_p) = \{0,1,p-1\}$
3. $itc(Z_p) = \{0,1,2,p-2,p-1\}$

Proof:

1. Determining $invo(Z_p)$, by definition: $invo(Z_p) = \{u \in Z_p \mid u^2 = 1\}$, in the field Z_p, the equation $u^2 = 1$ has exactly two solutions: $u = 1$ and $u = -1$, Since $-1 \equiv p-1$, we have: $invo(Z_p) = \{1, p-1\}$.
2. Determining $Tri(Z_p)$, by definition: $Tri(Z_p) = \{t \in Z_p \mid t^3 = t$, This equation can be rewritten as: $t^3 - t = t(t^2-1) = 0$, since Z_p is a field, we have: $t = 0$ or $t^2 = 1$, from part (1), $t^2 = 1$ gives $t = 1$ or $t = -1 \equiv p-1$. Therefore: $Tri(Z_p) = \{0,1,p-1\}$.
3. By definition: $itc(Z_p) = \{u + t \mid u \in invo(Z_p),\ t \in Tri(Z_p)\}$, we compute all possible sums:

u	t	$u+t$
1	0	1
1	1	2
1	$p-1$	$1+(p-1) = p \equiv 0$
$p-1$	0	$p-1$
$p-1$	1	$p \equiv 0$
$p-1$	$p-1$	$(p-1)+(p-1) = 2p-2 \equiv -2 \equiv p-2$

Collecting distinct elements modulo p: $itc(Z_p) = \{0,1,2,p-2,p-1\}$.■

Theorem 2.9:

For prime $p \geq 5$, the degree of the vertex v in $G_{itc}(Z_p)$ is: $\deg(v) = \begin{Bmatrix} 4 & if\ 2v \in itc(Z_p) \\ 5 & if\ 2v \notin itc(Z_p) \end{Bmatrix}$

Proof:

Let $v, w \in V(G_{itc}(Z_p))$, the vertex v is adjacent to the vertex w if $v + w \equiv g \in itc(Z_p)$, since $itc(Z_p) = \{0,1,2,p-2,p-1\}$, that is, the maximal number of adjacent vertices to v not exceed 5. Now for each $g \in itc(Z_p)$, there exists exactly one w such that $w \equiv g - v$, since the graph has no loops, we must exclude $w = v$: which implies that $v \equiv g - v$, $2v \equiv g$. If $2v \in itc(Z_p)$, one of neighbourhood of v it will v itself, so $\deg(v) = 4$. If $2v \notin itc(Z_p)$, $\deg(v) = 5$.■

Theorem 2.10:

$G_{itc}(Z_p)$ have exactly 5 vertices with degree 4 are: $\left\{0,1,\frac{p-1}{2},\frac{p+1}{2},p-1\right\}$, for prime $p \geq 5$.

Proof:

From Theorem 2.9, the vertex $v \in V(G_{itc}(Z_p))$ has degree 4 if $2v \in itc(Z_p)$, since $itc(Z_p) = \{0,1,2,p-2,p-1\}$, we solve the $2v \equiv g$ for each $g \in itc(Z_p)$, since $2 \in Z_p$ and Z_p is field so 2^{-1} is exists, $2.2^{-1} = 2.\frac{p+1}{2} =$

$p+1=1$.

For $0 \in itc(Z_p)$, $2v \equiv 0$, $2^{-1} \cdot 2v \equiv 2^{-1} \cdot 0$, $v=0$.
For $1 \in itc(Z_p)$, $2v \equiv 1$, $2^{-1} \cdot 2v \equiv 2^{-1} \cdot 1$, $v = 2^{-1} \equiv \frac{p+1}{2}$.
For $2 \in itc(Z_p)$, $2v \equiv 2$, $2^{-1} \cdot 2v \equiv 2^{-1} \cdot 2$, $v=1$.
For $p-2 \in itc(Z_p)$, $2v \equiv p-2$, $2^{-1} \cdot 2v \equiv 2^{-1} \cdot (p-2) = 2^{-1} \cdot (2p-2)$, $v = p-1$.
For $p-1 \in itc(Z_p)$, $2v \equiv p-1$, $2^{-1} \cdot 2v \equiv 2^{-1} \cdot (p-1)$, $v = \frac{p-1}{2}$.

The vertices have exactly degree five is $\left\{0,1,\frac{p-1}{2},\frac{p+1}{2},p-1\right\}$.■

Theorem 2.11:

The number of edges of $G_{itc}(Z_p)$ is $\frac{5(p-1)}{2}$, for prime $p \geq 5$.

Proof:

From Theorem 2.9, the degree of a vertex v in $G_{itc}(Z_p)$ is: $\deg(v) = \begin{Bmatrix} 4 & if\ 2v \in itc(Z_p) \\ 5 & if\ 2v \notin itc(Z_p) \end{Bmatrix}$, $\left|E\left(G_{itc}(Z_p)\right)\right| = \frac{1}{2}\left(\sum_{2v \in itc(Z_p)} \deg(v) + \sum_{2v \notin itc(Z_p)} \deg(v)\right)$, By Theorem 2.2, exactly 5 vertices have degree 4, $\left|V\left(G_{itc}(Z_p)\right)\right| = p$, so the remaining degree 5 is $p-5$.

$$\left|E\left(G_{itc}(Z_p)\right)\right| = \frac{1}{2}(5 \cdot 4 + (p-5) \cdot 5) = \frac{5(p-1)}{2}.■$$

Theorem 2.12:

The graph $G_{itc}(Z_p)$ is connected.

Proof

Let $v, w \in V\left(G_{itc}(Z_p)\right)$, If there are two distinct vertices, we must show that there exists a path between v and w.

Case 1: $v + w \in V\left(G_{itc}(Z_p)\right)$.

In this case, by the definition of the $G_{itc}(Z_p)$, there exists an edge $e = (v, w)$. Thus, a path of length 1 exists.

Case 2: $v + w \notin V\left(G_{itc}(Z_p)\right)$.

If the sum is not an invo-t-clean element, we must find an intermediate vertex or vertices make a path between v and w.

1. Consider the vertex $-x$ since $x + -x = 0 \in itc(Z_p)$, there is an edge between x and $-x$.
2. We can construct a path for x to $x-2$ using an intermediate vertex $1-x$.
   - $(x, 1-x)$ is edge, since $x + 1 - x = 1 \in itc(Z_p)$
   - $(1-x, x-2)$ is edge, since $1 - x + x - 2 = -1 \in itc(Z_p)$

   This creates a path of length two: $x \to 1-x \to x-2$, repeat the last two points for $x-2$.
   - $(x-2, 1-(x-2))$ is edge, since $x - 2 + 1 - (x-2) = 1 \in itc(Z_p)$
   - $(1-(x-2), x-4)$ is edge, since $1 - (x-2) + x - 4 = -1 \in itc(Z_p)$

   This creates a path of length four: $x \to 1-x \to x-2 \to 1-(x-2) \to x-4$
3. Since $\gcd(2,p) = 1$ for any odd prime p, the element 2 generates the cyclic group $(Z_p, +)$. Consequently, the sequence $\{x, 1-x, x-2, 1-(x-2), x-4, \dots, x-2k\}$, where k is a positive integer. Eventually contains all elements of Z_p. Thus, for any w, there is a path from v to w. ■

Theorem 2.13:

Let $p > 3$ be a prime number, and let $G_{itc}(Z_p)$ be the graph with vertex set Z_p, then the diameter of this graph is given by $\operatorname{diam}(G_{itc}(Z_p)) \geq \left\lfloor \frac{p+2}{4} \right\rfloor$.

Proof:

For each $s \in itc(Z_p)$, define the map $T_s: Z_p \to Z_p$ by $T_s(x) = -x + s$. Observe that x and y are adjacent precisely when $y = T_s(x)$ for some $s \in itc(Z_p)$. Thus, every edge corresponds to applying one such reflection map, and any path of length k from x to y corresponds to a composition

$$y = T_{s_k} \circ T_{s_{k-1}} \circ \cdots \circ T_{s_1}(x), s_i \in S.$$

A direct calculation shows: $T_{s_2} \circ T_{s_1}(x) = -(-x + s_1) + s_2 = x + (s_2 - s_1)$. Therefore:

- One reflection changes the sign: $x \mapsto -x + s$.
- Two reflections compose a pure translation: $x \mapsto x + d$, where $d = s_2 - s_1$.

Since each $s_i \in itc(Z_p) = \{0, \pm 1, \pm 2\}$, the translation distance satisfies $| d | \leq 4$. Consequently, every two steps in a path produce a translation of magnitude at most 4.

From the above, we deduce:

- After $2k$ steps, we can reach the vertices of the form $x + D$ where $| D | \leq 4k$.
- After $2k + 1$ steps, we can reach the vertices of the form $-x + D$ where $| D | \leq 4k + 2$.

The diameter equals the maximum distance from the vertex 0 to any other vertex: $\text{diam}(G_{itc}(Z_p)) = \max\limits_{a \in Z_p} d(0, a)$. In the cyclic group Z_p, the farthest vertex from 0 is $M = \left\lfloor \frac{p}{2} \right\rfloor = \frac{p-1}{2}$. Thus, $\text{diam}(G_{itc}(Z_p)) = d(0, M)$.

Lower bound for $d(0, M)$. We determine the minimum path length required to reach M from 0.

Case 1: Even-length paths

Suppose there exists a path of even length $2k$ from 0 to M. Then by the even-length property: $M \leq 4k$. Solving for k: $k \geq \frac{M}{4}$. Since k must be an integer, we have: $k \geq \left\lceil \frac{M}{4} \right\rceil$. Therefore, the path length satisfies: $2k \geq 2\left\lceil \frac{M}{4} \right\rceil$. Substituting $M = \frac{p-1}{2}$: $2\left\lceil \frac{p-1}{8} \right\rceil$.

This gives us a candidate lower bound from even paths.

Case 2: Odd-length paths

Suppose there exists a path of odd length $2k + 1$ from 0 to M. Then by the odd-length property:

$M \leq 4k + 2$. Solving for k: $4k \geq M - 2 \Rightarrow k \geq \frac{M-2}{4}$. Since k must be an integer: $k \geq \left\lceil \frac{M-2}{4} \right\rceil$. Therefore, the path length satisfies: $2k + 1 \geq 2\left\lceil \frac{M-2}{4} \right\rceil + 1$. Substituting $M = \frac{p-1}{2}$: $2\left\lceil \frac{p-5}{8} \right\rceil + 1$. This gives us a candidate lower bound from odd paths.

The actual distance $d(0, M)$ is the smallest path length that can reach M. This smallest length could be either even or odd, whichever is smaller. Therefore:

$$d(0, M) \geq \min\left\{2\left\lceil \frac{p-1}{8} \right\rceil, 2\left\lceil \frac{p-5}{8} \right\rceil + 1\right\}.$$

We now compute this minimum for all possible residues of p modulo 8. Since p is an odd prime greater than 3, the possible residues are 1, 3, 5, 7 modulo 8.

Let $p = 8\ell + r$ where $r \in \{1,3,5,7\}$.

$p \bmod 8$	Even bound	Odd bound	Minimum
1	2ℓ	$2\ell + 1$	2ℓ
3	$2\ell + 2$	$2\ell + 1$	$2\ell + 1$
5	$2\ell + 2$	$2\ell + 1$	$2\ell + 1$
7	$2\ell + 2$	$2\ell + 3$	$2\ell + 2$

Connecting to the desired formula, we now show that these minimum values equal $\left\lfloor \frac{p+2}{4} \right\rfloor$.

Compute $\left\lfloor \frac{p+2}{4} \right\rfloor$ for each case:

For $p = 8\ell + 1$: $\frac{8\ell+3}{4} = 2\ell + \frac{3}{4}$, floor = 2ℓ

For $p = 8\ell + 3$: $\frac{8\ell+5}{4} = 2\ell + \frac{5}{4}$, floor = $2\ell + 1$

For $p = 8\ell + 5$: $\frac{8\ell+7}{4} = 2\ell + \frac{7}{4}$, floor = $2\ell + 1$

For $p = 8\ell + 7$: $\frac{8\ell+9}{4} = 2\ell + \frac{9}{4}$, floor = $2\ell + 2$

These match exactly the minimum values we found.■

Proposition 2.14:

For any two vertices $x, y \in Z_p$:

$$d(x, y) \leq \left\lceil \frac{\min(| x - y |, p - | x - y |)}{2} \right\rceil \leq \left\lceil \frac{\lfloor p/2 \rfloor}{2} \right\rceil.$$

Proof:

From any vertex v, we can progress at most 2 units toward any target direction in a single step. From vertex $v \in V\big(G_{itc}(Z_p)\big)$, its neighbors are:

$$N(v) = \{-v,\ -v+1,\ -v-1,\ -v+2,\ -v-2\}$$

The fundamental move is $v \mapsto -v + k$ where $k \in \{0, \pm 1, \pm 2\}$.

In two consecutive steps: $v \overset{\text{step 1}}{\rightarrow} -v + k_1 \overset{\text{step 2}}{\rightarrow} -(-v + k_1) + k_2 = v - k_1 + k_2$

The net progress after two steps is: Progress $= | k_2 - k_1 | \leq | k_2 | + | k_1 | \leq 2 + 2 = 4$

Therefore, in two steps, the maximum progress is 4 units. Consequently, in m steps, the maximum progress is: Max Progress $\leq 2m$ units. Thus, in a single step, the maximum progress is 2 units.

The circular distance between x and y is: $\delta = \min(| x - y |, p - | x - y |)$, this represents the shortest distance around the cyclic group Z_p. Since each step covers at most 2 units of progress. Therefore, the number of steps required satisfies:

$$d(x,y) \leq \left\lceil \frac{\delta}{2} \right\rceil = \left\lceil \frac{\min(| x - y |, p - | x - y |)}{2} \right\rceil$$

Since $\delta \leq \lfloor p/2 \rfloor$ (the maximum circular distance), we have:

$d(x,y) \leq \left\lceil \frac{\lfloor p/2 \rfloor}{2} \right\rceil$, thus, $d(x,y) \leq \left\lceil \frac{\lfloor p/2 \rfloor}{2} \right\rceil$. ∎

Proposition 2.15:

Let $p > 3$ be a prime number, then $\left\lceil \frac{\lfloor p/2 \rfloor}{2} \right\rceil = \left\lfloor \frac{p+2}{4} \right\rfloor$.

Proof:

We consider the possible residues of p modulo 4. Since p is an odd prime greater than or equal to 5, we have $p \equiv 1 (\text{mod} 4)$ or $p \equiv 3 (\text{mod} 4)$.

Case 1: $p \equiv 1 (\text{mod} 4)$ (i.e., $p = 4k + 1$)

$$\lfloor p/2 \rfloor = \lfloor (4k+1)/2 \rfloor = 2k$$
$$\left\lceil \frac{2k}{2} \right\rceil = \lceil k \rceil = k$$
$$\left\lfloor \frac{p+2}{4} \right\rfloor = \left\lfloor \frac{4k+3}{4} \right\rfloor = \left\lfloor k + \frac{3}{4} \right\rfloor = k$$

Thus, both sides equal k.

Case 2: $p \equiv 3 (\text{mod} 4)$ (i.e., $p = 4k + 3$)

$$\lfloor p/2 \rfloor = \lfloor (4k+3)/2 \rfloor = 2k + 1$$
$$\left\lceil \frac{2k+1}{2} \right\rceil = \left\lceil k + \frac{1}{2} \right\rceil = k + 1$$
$$\left\lfloor \frac{p+2}{4} \right\rfloor = \left\lfloor \frac{4k+5}{4} \right\rfloor = \left\lfloor k + \frac{5}{4} \right\rfloor = k + 1$$

Thus, both sides equal $k + 1$.

Case 3: $p \equiv 0 (\text{mod} 4)$ — impossible for primes $p \geq 5$

Case 4: $p \equiv 2 (\text{mod} 4)$ — impossible for primes $p \geq 5$ except $p = 2$

Therefore, for all primes $p > 3$: $\left\lceil \frac{\lfloor p/2 \rfloor}{2} \right\rceil = \left\lfloor \frac{p+2}{4} \right\rfloor$. ∎

Theorem 2.16:

Let $p > 3$ be a prime number, and let $G_{itc}(Z_p)$ be the graph with vertex set Z_p, then the diameter of this graph is given by $\text{diam}(G_{itc}(Z_p)) \leq \left\lfloor \frac{p+2}{4} \right\rfloor$. ∎

Proof:

By definition, the diameter is: $\text{diam}(G) = \max_{x,y \in V} d(x,y)$

From Proposition 2.14, for any $x, y \in V: d(x,y) \leq \left\lceil \frac{\lfloor p/2 \rfloor}{2} \right\rceil$, therefore: $\text{diam}(G) = \max_{x,y \in V} d(x,y) \leq \left\lceil \frac{\lfloor p/2 \rfloor}{2} \right\rceil$

From Proposition 2.15: $\left\lceil \frac{\left\lfloor \frac{p}{2} \right\rfloor}{2} \right\rceil = \left\lfloor \frac{p+2}{4} \right\rfloor$. $\text{diam}\left(G_{itc}(Z_p)\right) \leq \left\lfloor \frac{p+2}{4} \right\rfloor$. ■

Corollary 2.17:

Let $p > 3$ be a prime number, and let $G_{itc}(Z_p)$ be the graph with vertex set Z_p, then the diameter of this graph is given by $\text{diam}\left(G_{itc}(Z_p)\right) = \left\lfloor \frac{p+2}{4} \right\rfloor$. ■

Example 2.18:

Find the lower bound diameter for the graph $G_{itc}(Z_{23})$ using Theorem 2.13.

- $p = 23$
- $itc(Z_{23}) = \{0,1,2,-1,-2\} = \{0,1,2,22,21\}$ in Z_{11}
- Farthest vertex from 0: $M = \frac{p-1}{2} = \frac{22}{2} = 11$

Define reflections $T_s(x) = -x + s$ for each $s \in itc(Z_{11})$. Adjacency: $x \sim y \iff y = T_s(x)$ for some $s \in itc(Z_{23})$.

$T_{s_2} \circ T_{s_1}(x) = x + (s_2 - s_1)$

- $T_2 \circ T_{-2}(x) = x + (2 - (-2)) = x + 4$
- $T_1 \circ T_{-1}(x) = x + (1 - (-1)) = x + 2$
- $T_0 \circ T_1(x) = x + (0 - 1) = x - 1$

Even and Odd Length Paths

- After $2k$ steps: can reach D with $| D | \leq 4k$
- After $2k + 1$ steps: can reach D with $| D | \leq 4k + 2$

Reduction to Distance from 0: Due to vertex-transitivity, the diameter equals $d(0, M) = d(0,11)$.

Lower Bound for $d(0,11)$

Even case ($2k$ steps): $11 \leq 4k \Rightarrow k \geq \frac{11}{4} = 2.75 \Rightarrow k \geq 3 \Rightarrow 2k \geq 6$

Odd case ($2k + 1$ steps): $11 \leq 4k + 2 \Rightarrow 4k \geq 9 \Rightarrow k \geq \frac{9}{4} = 2.25 \Rightarrow k \geq 3 \Rightarrow 2k + 1 \geq 7$

The minimum of these lower bounds is $\min(6,7) = 6$.

Compute $\left\lfloor \frac{p+2}{4} \right\rfloor = \left\lfloor \frac{23+2}{4} \right\rfloor = \left\lfloor \frac{25}{4} \right\rfloor = 6$. Thus $d(0,11) \geq 6$.

Path: $0 \to 21 \to 4 \to 17 \to 8 \to 13 \to 11$

Check each edge:

1. $0 + 21 = 21 \in itc(Z_{23})$
2. $21 + 4 = 25 \equiv 2 \in itc(Z_{23})$
3. $4 + 17 = 21 \in itc(Z_{23})$
4. $17 + 8 = 25 \equiv 2 \in itc(Z_{23})$
5. $8 + 13 = 21 \in itc(Z_{23})$
6. $13 + 11 = 24 \equiv 1 \in itc(Z_{23})$

Path length = 6 steps.

$\text{diam}(G_{itc}(Z_{23})) \geq 6$

Find the upper bound diameter of the graph $G_{itc}(Z_{23})$ using Proposition 2.13 and Theorem 2.16.

$d(0,11) \leq \left\lceil \frac{\delta}{2} \right\rceil = \left\lceil \frac{\min\{|0-11|, 23-|0-11|\}}{2} \right\rceil = \left\lceil \frac{\min(11,12)}{2} \right\rceil = 6$, also $\text{diam}\left(G_{itc}(Z_{23})\right) \leq \left\lceil \frac{\left\lfloor \frac{23}{2} \right\rfloor}{2} \right\rceil = \left\lceil \frac{12}{2} \right\rceil = 6$,

$\text{diam}(G_{itc}(Z_{23})) \leq 6$. Therefore $\text{diam}\left(G_{itc}(Z_{23})\right) = 6$.

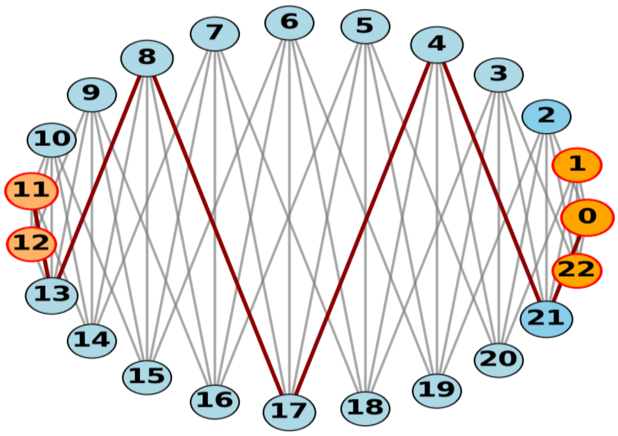

Figure 5: The Graph $G_{itc}(Z_{23})$.

Conclusions:

This study examines the structure of a graph defined on the basis of invo-t-clean elements within rings of integers modulo a prime number. The paper challenge inherent in this new definition, compared to previous definitions that yielded fixed graphs, lies in the fact that the graphs in our study exhibit notable structural variation as the prime number changes. Nevertheless, we successfully identified the fundamental properties common to these graphs. We proved that all resulting graphs are connected and uniquely contain exactly five vertices of degree four, corresponding precisely to the number of invo-t-clean elements in the ring and the other of degree five. Furthermore, we were able to determine the graph's diameter, a task that proved challenging due to its direct and variable dependence on the value of the prime number

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Data Availability

There is no data in this paper.

Conflicts of Interest

The authors called for a lack of conflicts of interest to publish this paper.

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