Complementary Topological Graphs: Structural Properties and Domination Parameters

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Abstract:

In this thesis, a new type of graph called the complementary topological graph, denoted by G_c , is introduced and studied. This graph is constructed from a finite set X equipped with the discrete topology, where the vertices represent all non-empty proper subsets of X, and two vertices are adjacent whenever their union equals the whole set X . The work focuses on studying several structural properties ofG_c , including the order, size, degree of vertices, Eulerian properties, and chromatic number. It is shown that the graph contains $2^n - 2$ vertices and $\frac{3^n-2^{n+1}+1}{2}$ edges, where $|X| = n$. In addition, the graph is proved to be regular with degree $2^{n-1} - 1$. Based on these properties, it is concluded that G_c is neither Eulerian nor semi-Eulerian for all $n \geq 3$. The chromatic number of the graph is also determined and shown to be equal to n.

Furthermore, this thesis investigates domination in complementary topological graphs under some graph operations, particularly the corona and join operations. The domination number for these operations is obtained and characterized through several results and examples. The results presented in this thesis provide a deeper understanding of the relationship between graph theory and topology, and they may serve as a basis for future studies on complementary topological graphs and their applications in different areas of graph theory.

Keywords: Complementary Topological Graphs, Domination Number, Graph Operations, Join Operation, Corona Operation, Chromatic Number, Eulerian Graphs, Semi-Eulerian Graphs, Discrete Topology, Graph Theory.

Introduction:

Graph Theory is one of the important branches of mathematics that studies the relationships between objects through vertices and edges. In recent years, the interaction between graph theory and Topology has attracted considerable attention, leading to the construction of several classes of topological graphs with different structural and domination properties. One of the important concepts in graph theory is graph coloring. The chromatic number of a graph G, denoted by $\chi(G)$, is the minimum number of colors required to color the vertices of G such that no two adjacent vertices receive the same color. Another important concept is Eulerian properties. A connected graph is called Eulerian if every vertex has even degree, while it is called semi-Eulerian if exactly two vertices have odd degree. Graph operations also play a fundamental role in studying graph structures and domination properties. Among these operations are the join and corona operations. The join $G_1 + G_2$ of two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is obtained by joining every vertex of G_1 to every vertex of G_2. On the other hand, the corona $G_1 \odot G_2$ is constructed by taking one copy of G_1 together with copies of G_2, and connecting each vertex of G_1 to all vertices in its corresponding copy of G_2. The main aim of this paper is to study the structural and domination properties of complementary topological graphs. In particular, the order, size, chromatic number, Eulerian properties, and domination numbers under join and corona operations are investigated.

Definition of Complementary Topological Graph

Definition 1: Let X be a non-empty set and τ be a discrete topology on X. The complementary topological graph denoted by $G_c = (V, E)$ is a graph of the vertex set $V = \{A;\ A \in \tau$ and $A \neq ø, X\}$, and the edge set $E = \{A\,B;\ A \cup B = X$, where $A, B \in \tau\}$.

Proposition 1.1: The order of complementary topological graph G_c is $2^n - 2$, for $|X| = n$.

Proof: Since τ be a family of all subsets of X which are 2^n sets. Then, G_c has all elements of τ unless ø and X by Definition 2.2.1

Theorem 1.2: Let X be a finite set with $|X| = n$ and let τ be the discrete topological space. Then, the number of edges of G_c, is $|E(G_c)| = \frac{3^n - 2^{n+1} + 1}{2}$.

Proof: Assume $n = 2$, then, the order of edges is one which are two vertices. There is exactly one edge between them.
Also, if $n = 3$, $|E(G_c)| = \frac{3^3 - 2^{3+1} + 1}{2} = \frac{27-16+1}{2} = \frac{12}{2} = 6$

when $n = 4$. The number of edges is $|E(G_c)| = \frac{3^4 - 2^{4+1} + 1}{2} = \frac{81-32+1}{2} = \frac{50}{2} = 25$. We can note that in figure

The computed values agree with the given formula for the tested cases. This supports the correctness of the formula for the number of edges of G_c .

$n = 2$, then, the order of edges is one. If $n = 3$, we note the order of edges is 6, when $n = 4$. The number of edges is 25.

Proposition1.3: The complementary topological graph G_c is non-Eulerian and not semi-Eulerian

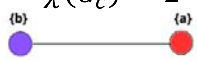
Proof. Let $|X| = n \geq 3$ and let G_c be the complementary topological graph. From Theorem 2.7, for every vertex $A \in V(G_c)$ deg $G_c(A) = 2^{n-1} - 1$. Hence, G_c is a regular graph of degree $2^{n-1} - 1$. Since $2^{n-1} - 1$ is even for all $n \geq 2$, it follows that $2^{n-1} - 1$ is odd. every vertex of G_c has odd degree. Moreover, G_c is connected. By Eulers theorem, a connected graph is Eulerian if and only if all its vertices have even degree, and it is semi- Eulerian if and if exactly two vertices have odd degree. Since in G_c all vertices have odd degree and their number exceeds two for $n \geq 3$, it follows that G_c is neither Eulerian nor semi-Eulerian. Therefore G_c is non-Eulerian for all $n \geq 3$.

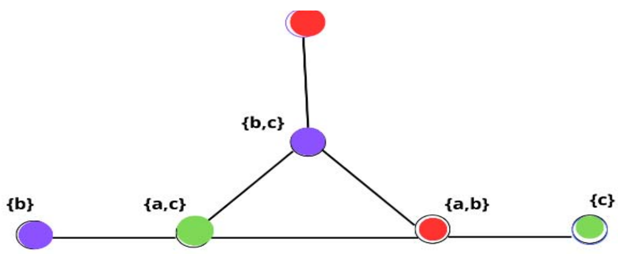
Theorem1.4: Let X be a finite set with $|X| = n \geq 2$, and let τ be the discrete topology on X. Then the chromatic number of G_c satisfies $\chi(G_c) = n$.

Proof: Assum that $n = 2$. Since the two vertices are adjacent, they cannot be assigned the same color. Hence $\chi(G_c) = 2$. When $n = 3$ then the pendent vertices its singleton vertex is adjacent with complementary. Therefore, every vertex must receive a color different from the color assigned to its complement. Moreover, the complementary vertices are also connected according to the structure shown in the graph. When $n = 5$ Since the vertices adjacent with complementary. The singleton vertex adjacent with the complementary vertices and this vertex is connected $\chi(G_c) = n$. The chromatic number $\chi(G_c) = n$ because the graph contains a complete induced subgraph of order n (K_n). Hence, at least n colors are required.

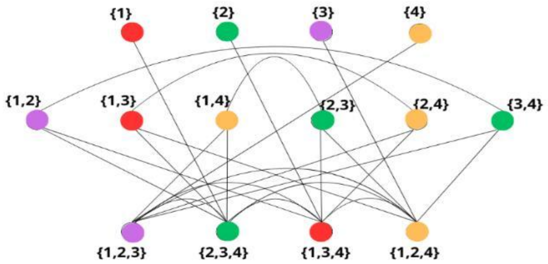
Example 1.4: We will apply Theorem 2.2.24 in the graph G_c if $|X| = 2$

$\chi(G_c) = 2$

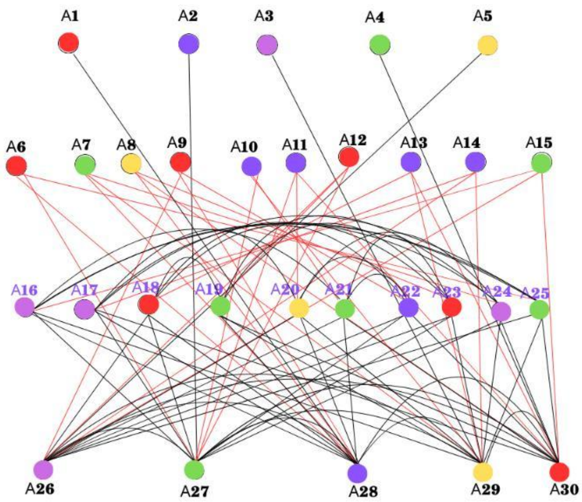




$$\chi(G_c) = 3$$



$$\chi(G_c) = 4$$

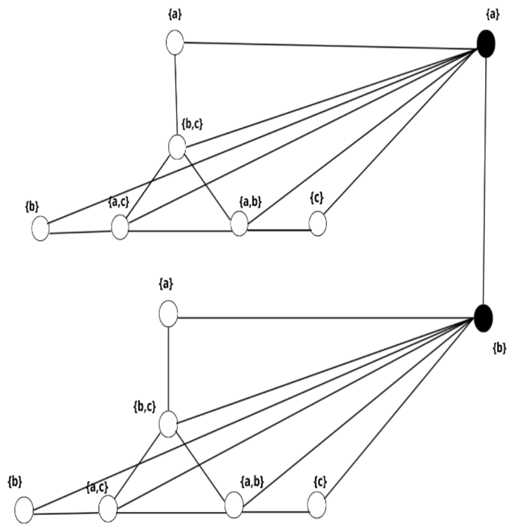


$$\chi(G_c) = 5$$

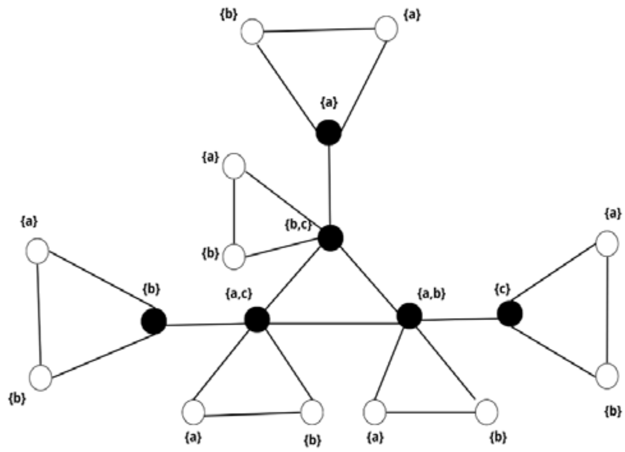
Figure 1: The chromatic number

Theorem 3.2.5: Let G_c be a complementary topological graph defined on a set X, and R_c be a complementary topological graph defined on a set W such that $|X| = n$ and $|W| = m$. Then, $G_c \odot R_c$ has dominating set and domination number where: $\gamma(G_c \odot R_c) = 2^n - 2$.

Proof: By definition of corona operation $G_c \odot R_c$, each vertex in G_c is adjacent with all vertices of one copy of a graph R_c. This means if $v_i \in V(G_c)$, then v_i is adjacent with all vertices of the i^{th} copy of R_c. Thus, v_i dominates all vertices of one copy of R_c, so that $v_i \in D$. Then, $D = V(G_c)$ is the minimum dominating set of a graph $G_c \odot R_c$. Since the number of all vertices in a graph G_c is $2^n - 2$ according to Proposition 2.2.19. Thus, the order of minimum dominating set D is $2^n - 2$. Therefore, $\gamma(G_c \odot R_c) = 2^n - 2$. As an example, see Figure 3.5.



(a) $k_2 \odot \overline{C_6}$

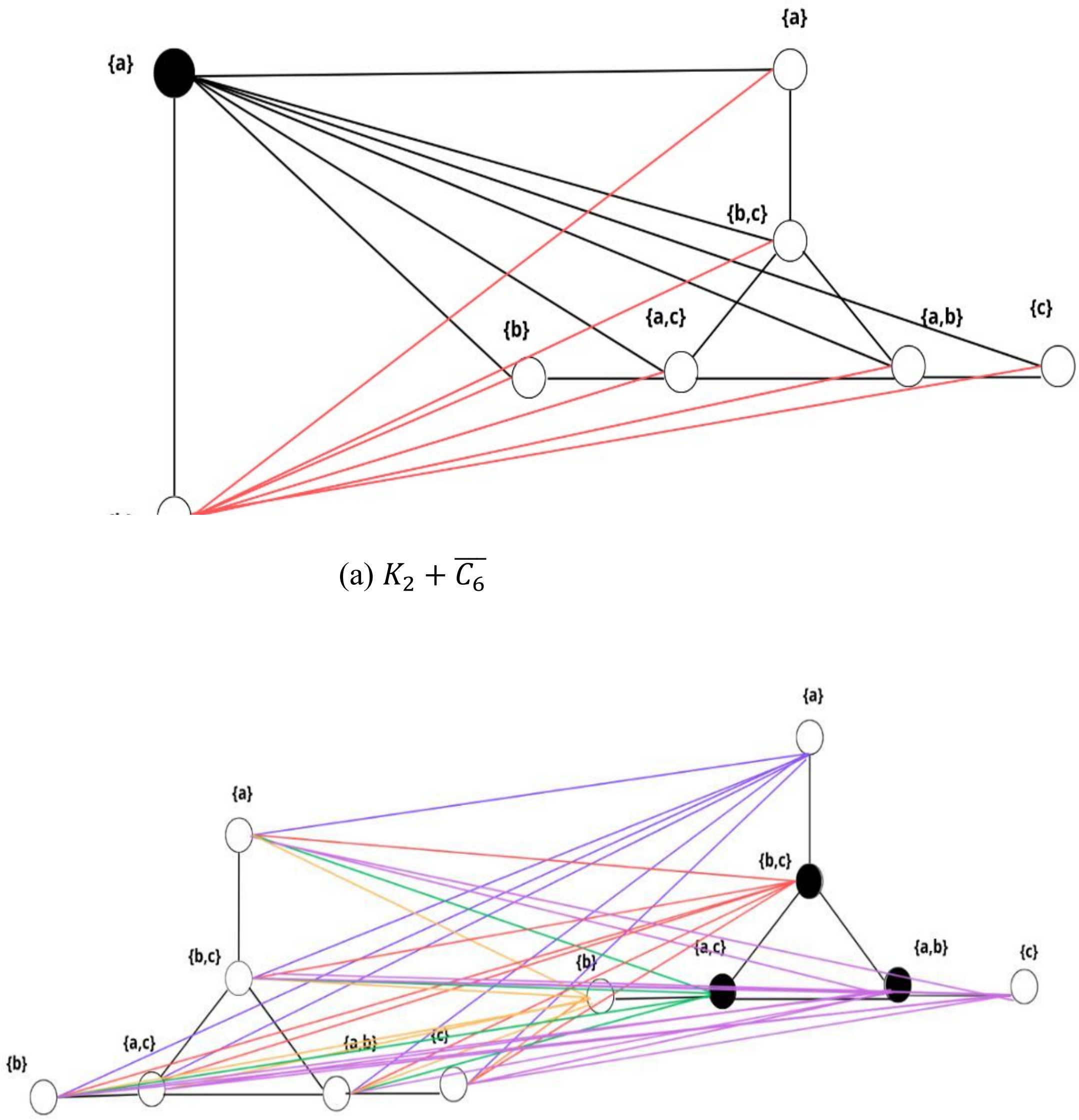


(b)Figure $\overline{C_6} \odot k_2$

Figure 2: A minimum dominating set in $k_2 \odot \overline{C_6}$ and $\overline{C_6} \odot k_2$.

Theorem 1.6: Let G_c be a discrete topological graph defined on a set X, and R_c be a complementary topological graph defined on a set W. If $|X| = n$ and $|W| = m$, then $\gamma\ (G_c + R_c) =$
$$\begin{cases} 1, & if \quad n = 2 \; or \; m = 2 \\ n, & if \quad n, m > 2 \end{cases}$$

Proof: In the graph $G_c + R_c$ each vertex from a graph G_c is adjacent to all vertices of a graph R_c according to the definition of join operation. Then, if $n = 2$ and from proof Theorem 3.2.1. The graph G_c contains one dominating vertex say v and this vertex will dominates all vertices of a graph R_c and one vertex of G_c (the proof is similar if $m = 2$. Then, D is a minimum dominating set of a graph $G_C + R_c$ and $\gamma\ (G_c + R_c) = 1$. Now if $n, m\ > 2$ and from proof of Theorem 3.2.1. The graph G_c has n dominating set vertices say$\{u_1, u_2, \dots \dots u_n\}$ have $(n - 1)$ elements where $\gamma\ (G_c) = n$. Since $\{u_1, u_2, \dots \dots u_n\}\ \in V\ (G_c)$. Then, each vertex of a set $D = (u_1, u_2, \dots \dots u_n)$ is adjacent to all vertices of the graph (R_s) according to the definition of join operation. So that, $D = \{u_1, u_2, \dots \dots u_n\}$ be a minimum dominating set in a graph $G_c + R_c$. Thus, $\gamma(G_c + R_c) = n$



(a) $K_2 + \overline{C_6}$

(b) $\overline{C_6}$ $+\overline{C_6}$

Figure 3: A minimum dominating set in join operation graphs.

Conclusion:

In this thesis, a new graph structure called the complementary topological graph $G_c$was introduced and studied on a finite discrete topological space. The construction of this graph depends on the relation between subsets of a finite set X, where two vertices are adjacent whenever their union equals X. Several structural and domination properties of $G_c$were investigated and characterized.

The study determined important graph parameters such as the order, size, degree of vertices, Eulerian properties, and chromatic number. It was proved that the order of G_c is $2^n - 2$ while the number of edges is given by $\frac{3^n - 2^{n+1} + 1}{2}$. Moreover, the graph was shown to be regular of degree $2^{n-1} - 1$. Based on these results, it was concluded that G_c is neither Eulerian nor semi-Eulerian for all $n \geq 3$. In addition, the chromatic number of the graph was established as $\chi(G_c) = n$.

The domination behavior of complementary topological graphs under graph operations was also studied. In particular, the domination number for corona and join operations was obtained and characterized. The results demonstrated that graph operations significantly affect domination properties and lead to different domination structures.

The obtained results contribute to the interaction between graph theory and topology, especially in the study of graphs constructed from topological spaces. This work opens the door for future investigations concerning other graph parameters such as metric dimension, resolving sets, Hamiltonicity, planarity, and different types of domination on complementary topological graphs and their related operations

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